9

Exponential vs. Linear Growth

The concepts in this chapter build on what we learned earlier in the percents chapter. Let's see how.

EXAMPLE 1: Jonas has a savings account that earns 3 percent interest compounded annually. His initial deposit was \$800. Which of the following expressions gives the value of the account, in dollars, after 10 years?

A) $800(1.30)^{10}$ B) $800 + 30(10)$ C) $800(1.03)(10)$ D) $800(1.03)^{10}$

A 3 percent interest rate compounded annually means Jonas earns 3 percent on the account once a year. This isn't just 3% on the original \$800. This is 3% on whatever's in his account at the time, including any interest he's already earned in previous years. So if we're in year 5, Jonas would earn 3% interest on the original \$800 and 3% interest on the total interest accumulated in years one through four. This interest that's earned on top of interest is what defines **compound interest**.

If we try to calculate the interest earned in the account after each and every year, this problem would take forever. Instead, let's take what we learned from the percents chapter and apply it here. As we know, a 3% increase is equivalent to a multiplication factor of 1.03. Therefore, the total value of the account after each year is just 1.03 times whatever the value was last year:

End of Year 1 total: $800(1.03) = 800(1.03)^1$
End of Year 2 total: $800(1.03)(1.03) = 800(1.03)^2$
End of Year 3 total: $800(1.03)(1.03)(1.03) = 800(1.03)^3$
End of Year 4 total: $800(1.03)(1.03)(1.03)(1.03) = 800(1.03)^4$

See the pattern in the exponents? The value of the account after t years is just $800(1.03)^t$. Therefore, the value of the account after 10 years is $800(1.03)^{10}$, answer $\boxed{(D)}$. No calculations needed.

EXAMPLE 2: The population of an ant colony doubles every month. If there were 500 ants in the colony on May 1st, which function represents the population of the ant colony, $A(t)$, t months after May 1st?

A) $A(t) = 500(2t)$ B) $A(t) = 2(500)^t$ C) $A(t) = 500 + 2t$ D) $A(t) = 500(2)^t$

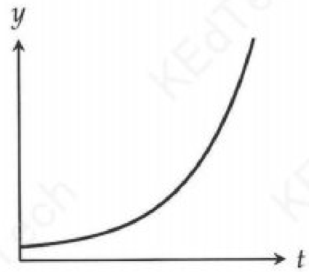
Since the population doubles every month, the population after the first month is 500(2). The population after the second month is double the population after the first month: $500(2)(2) = 500(2)^2$. The population after the third month is double the population after the second month: $500(2)(2)(2) = 500(2)^3$. See where this is going? The pattern is exactly the same as in Example 1 except the initial value here is 500 and the multiplication factor is 2. Therefore, the ant population after t months is $500(2)^t$, answer $\boxed{(D)}$.

A bank account earning 3 percent every year and an ant population doubling every month are examples of **exponential growth**, which occurs when a value grows periodically by a factor greater than 1. As we've seen, exponential growth can be modeled as an equation that looks like

$$y = ab^t$$

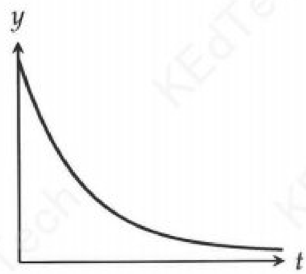
where y is the value after t specified units of time (e.g. years), a is the initial value, and b is a growth factor greater than one. The initial value a is sometimes referred to as the **coefficient** and the growth factor b is sometimes referred to as the **base**. In our bank account example, 800 was the initial value and 1.03 was the growth factor. In our ant population example, 500 was the initial value and 2 was the growth factor.

Graphs of exponential growth have the following shape:



Notice how the graph creeps up slowly at first but then shoots up faster and faster over time. That's exponential growth.

Exponential decay, however, is the opposite. Imagine a radioactive substance that loses mass over time. It loses a lot of its mass in the beginning and then loses it more and more slowly as time goes by.



It's worth memorizing the shapes of these graphs of exponential growth and decay. The SAT may test you explicitly on them.

The equation for exponential decay is the same as the equation for exponential growth:

$$y = ab^t$$

The only difference is that the growth factor, b, is less than 1. So an equation that models exponential decay might look like

$$y = 400(0.6)^t$$

where y is the mass, in grams, of a radioactive substance t years from now. The 400 indicates that the substance currently has a mass of 400 grams, and the 0.6 indicates that at the end of each year, the substance is left with 60% of the mass it started the year with. In other words, the substance loses 40% of its mass each year.

So far, the examples we've discussed have been relatively simple. To model more complicated cases of exponential growth and decay, such as a bank account growing by 3% every 2 years or a radioactive substance losing half its mass every 9 months, we'll need to use a more generalized equation, one that you absolutely need to memorize:

$$y = ab^{\frac{t}{k}}$$

where y is the value after t specified units of time, a is the initial value, and k is the time required for the value of y to change by a factor of b. Note that t and k must be in the same units. So if t is in years, k should also be in years.

Let's go over some examples so that you fully understand what k means.

EXAMPLE 3:

$$M(t) = 400(1.05)^{\frac{t}{3}}$$

The function M models the mass, in nanograms, of a particle after t years. Which of the following is the best interpretation of $(1.05)^{\frac{t}{3}}$ in this context?

A) The mass of the particle increases by 5% every 4 months.
B) The mass of the particle increases by 5% every 3 years.
C) The mass of the particle increases by 5 nanograms every 4 months.
D) The mass of the particle increases by 5 nanograms every 3 years.

We have an exponential equation with $k = 3$ and $b = 1.05$, which means it takes 3 years for the mass of the particle to change by a factor of 1.05. In other words, the particle's mass increases by 5% every 3 years. And indeed, if we plug in $t = 3$, we get $400(1.05)^1 = 400(1.05)$, which represents one 5% increase after 3 years. Answer $\boxed{(B)}$.

Let's see why the generalized equation $y = ab^{\frac{t}{k}}$ works by taking a deeper look at this example. The mass of the particle changes by a factor of 1.05 for every increase of 1 in the value of the exponent, $\frac{t}{3}$. If we start the value of the exponent at 0, then $\frac{t}{3} = 0$. Multiplying both sides by 3 yields $t = 0$ years. Now if we set the value of the exponent to 1, then $\frac{t}{3} = 1$. Multiplying both sides by 3 yields $t = 3$ years. Since the value of the exponent increases by 1 from $t = 0$ years to $t = 3$ years, 3 years must be the time required for the particle's mass to change by a factor of 1.05. Repeat these steps with k instead of 3, and you'll have proven why k represents the time required for the value of y to change by a factor of b in the generalized equation.

EXAMPLE 4:

$$M(t) = 400(0.6)^{3t}$$

The function M models the mass, in nanograms, of a particle after t years. Which of the following is the best interpretation of $(0.6)^{3t}$ in this context?

A) The mass of the particle decreases by 60% every 4 months.
B) The mass of the particle decreases by 60% every 3 years.
C) The mass of the particle decreases by 40% every 4 months.
D) The mass of the particle decreases by 40% every 3 years.

With $y = ab^{\frac{t}{k}}$ as a reference, we see that the exponent of the given equation is not in the form of $\frac{t}{k}$, so how are we supposed to find the value of k ? We have to use a little arithmetic trick:

$$M(t) = 400(0.6)^{3t} = 400(0.6)^{t/(1/3)}$$

Multiplying by 3 is the same as dividing by $\frac{1}{3}$. Now we can see that $k = \frac{1}{3}$, which means that the particle's mass changes by a factor of 0.6 every $\frac{1}{3}$ year. That's equivalent to a $1 - 0.6 = 40\%$ decrease every 4 months. Answer $\boxed{(C)}$. Note that this is an example of exponential decay.

EXAMPLE 5: A non-profit organization currently has 50 volunteers. If the organization is able to double the number of volunteers every 8 months, which of the following equations best models the number of volunteers, v, the organization will have t months from now?

A) $v = 50(2)^{\frac{t}{8}}$ B) $v = 50(2)^{\frac{t}{4}}$ C) $v = 50(2)^{t}$ D) $v = 50(2)^{8t}$

Again, we'll use $y = ab^{\frac{t}{k}}$ as a reference. Based on the given information, $a = 50$ and $b = 2$. Since the number of volunteers increases by a factor of 2 every 8 months, and t is in months, $k = 8$. Therefore, the equation that best models the number of volunteers is $v = 50(2)^{\frac{t}{8}}$. Answer $\boxed{(A)}$.

EXAMPLE 6: A non-profit organization currently has 50 volunteers. If the organization is able to double the number of volunteers every 8 months, which of the following equations best models the number of volunteers, v, the organization will have t years from now?

A) $v = 50(2)^{\frac{t}{8}}$ B) $v = 50(2)^{\frac{2t}{3}}$ C) $v = 50(2)^{\frac{3t}{2}}$ D) $v = 50(2)^{8t}$

This question is the same as the one in Example 3, except we now have inconsistent units of time. We again have a growth factor of 2, and since k (from our reference equation) is the amount of time it takes for a single change by the growth factor, we might assume that $k = 8$. However, this answer would be wrong because k is in months when the t in our equation needs to be in years.

To form the correct equation, t and k must be in the same units, so we have to convert 8 months into years: $k = \frac{8}{12} = \frac{2}{3}$ years. The equation is then $v = 50(2)^{t/(2/3)} = 50(2)^{\frac{3t}{2}}$. Answer $\boxed{(C)}$.

EXAMPLE 7:

$$f(m) = 25.5(1.72)^m$$

The given function f models the temperature of a substance m minutes after it is exposed to sulfur dioxide, where $0 \leq m \leq 10$. Which of the following functions best models the temperature of the substance s seconds after its exposure to sulfur dioxide, where $0 \leq s \leq 600$?

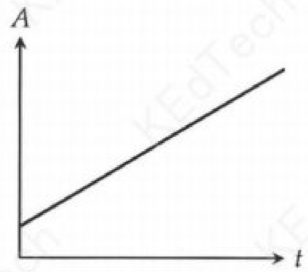
A) $g(s) = 25.5\left(\frac{1.72}{60}\right)^s$ B) $g(s) = 25.5\left(\frac{1.72}{60}\right)^{\frac{s}{60}}$ C) $g(s) = 25.5(1.72)^{\frac{s}{60}}$ D) $g(s) = 25.5(1.72)^{60s}$

The given function f tells us that the temperature of the substance increases by a factor of 1.72 (72% increase) every minute. That's the same as a 1.72 factor increase every 60 seconds. This conversion to seconds is important—because s is in seconds, the "k-value" has to be converted to seconds. So which of the answer choices has a growth factor of 1.72 and a k-value of 60? Only answer $\boxed{(C)}$.

Now let's compare exponential growth and decay with linear growth and decay. As you may already know, **linear growth** can be modeled by a line with a positive slope. For example, if Ann has a piggybank with 100 dollars already in it, and she adds 5 dollars every month, the total amount in the piggybank can be modeled by

$$A = 5t + 100$$

where A is the total amount, t is the number of months, and 100 (the y-intercept) is the initial amount.

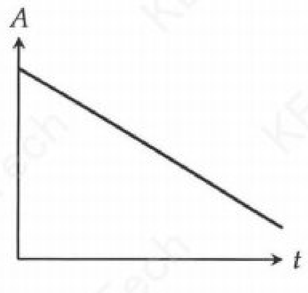


Unlike exponential growth, **linear growth** is constant and consistent. There is no slowing down or speeding up. The total goes up by the same amount each time.

Now imagine that Ann takes 5 dollars out of her piggybank every month instead of adding to it. The total balance would decrease by a constant amount each month, resulting in **linear decay**. The total amount A in the piggybank could then be modeled by

$$A = 100 - 5t$$

The graph of such an equation is a line with a negative slope:



EXERCISE 1: Answers for this exercise start on page 344.

1

Albert has a large book collection. He decides to trade in two of his used books for one new book each month at a local bookstore. Which of the following best describes the relationship between time (in months) and the total number of books in Albert's collection?

A) Increasing linear
B) Decreasing linear
C) Exponential growth
D) Exponential decay

2

A scientist counts 80 cells in a petri dish and finds that each one splits into two new cells every hour. He uses the function $A(t) = cr^t$ to calculate the total number of cells in the petri dish after t hours. Which of the following assigns the correct values to c and r ?

A) $c = 40, r = 2$
B) $c = 80, r = 0.5$
C) $c = 80, r = 1.5$
D) $c = 80, r = 2$

3

The number of deer in a population is expected to decrease by 11% each year. If the number of deer in the population is currently 12,000, which of the following is closest to the expected number of deer in the population 3 years from now?

A) 8,040
B) 8,460
C) 8,770
D) 9,320

4

Two variables, x and y, are related such that the value of y decreases by 25% for every increase in the value of x by 1. If $y = 4$ when $x = 0$, which of the following equations represents this relationship?

A) $y = 0.75(4)^x$
B) $y = 4(0.75)^x$
C) $y = 4(1.25)^x$
D) $y = 16(0.25)^{x+1}$

5

The employees at a new bookstore must stock a certain number of shelves so that the store is ready for its opening in two weeks. The employees stock shelves at a constant rate throughout the two weeks. If $p(t)$ is the number of shelves left to be stocked after t days, which of the following statements best describes the function p ?

A) The function p is an increasing exponential function.
B) The function p is a decreasing exponential function.
C) The function p is an increasing linear function.
D) The function p is a decreasing linear function.

6

The initial value of a car is \$10,000. The function v estimates that the value of the car at the end of each year is 65% of its value at the end of the previous year. Which equation defines v, where $v(k)$ is the estimated value, in dollars, of the car at the end of the kth year?

A) $v(k) = 10{,}000(0.35)^k$
B) $v(k) = 10{,}000(0.65)^k$
C) $v(k) = 10{,}000(1.35)^k$
D) $v(k) = 10{,}000(1.65)^k$

7

The function $p(t) = 13(1.1)^t$ models the population, in thousands, of a certain town from 2008 to 2020, where t represents the number of years after 2008. Which of the following is the best interpretation of "the value of $p(4)$ is approximately 19" in this context?

A) The population of the town grew by approximately 19 percent each year from 2008 to 2012.

B) The population of the town grew by approximately 19,000 from 2008 to 2012.

C) The population of the town was approximately 19,000 in 2012.

D) The population of the town grew by approximately 19,000 every 4 years between 2008 and 2020.

8

$$f(t) = 20\left(1 + \frac{15}{100}\right)^t$$

The function f models the temperature, in degrees Celsius, of a metal alloy used in an experiment, where t is the number of seconds after the experiment began. Which of the following is the best interpretation of the number 15 in this context?

A) The temperature, in degrees Celsius, of the metal alloy at the beginning of the experiment

B) The increase in the temperature, in degrees Celsius, of the metal alloy every 100 seconds during the experiment

C) The percent by which the temperature, in degrees Celsius, of the metal alloy decreased from each second to the next during the experiment

D) The percent by which the temperature, in degrees Celsius, of the metal alloy increased from each second to the next during the experiment

9

The equation $m(t) = 31.6(1.278)^t$ gives the estimated number of members, in thousands, of a wholesale club for $0 \leq t \leq 30$, where t is the number of years after 1976. Which of the following is the best interpretation of the number 31.6 in this context?

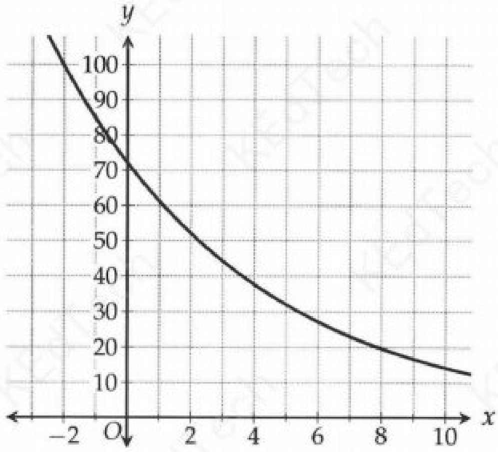
A) The estimated number of members, in thousands, in 1976

B) The estimated number of members, in thousands, t years after 1976

C) The estimated increase in the number of members, in thousands, each year after 1976

D) The estimated percent by which the number of members increased from each year to the next after 1976

10



The graph of $y = f(x)$ is shown, where the function f is defined by $f(x) = a(b)^x$, and a and b are positive constants. Which of the following is closest to the value of $\frac{a}{b}$?

A) 63

B) 85

C) 102

D) 480

11

Omar currently holds a government bond that has a market value of \$900. Each year, the market value of the bond is expected to be 20% greater than its market value the year before. If the expression $900(1+p)$, where p is a constant, represents the expected market value, in dollars, of the bond after 3 years, what is the value of p ?

12

Which of the following equivalent forms of the exponential function f displays the value of $f(4)$ as the base or the coefficient?

A) $f(x) = 2\left(\frac{1}{2}\right)^{x}$

B) $f(x) = 2\left(\frac{1}{4}\right)^{\frac{x}{2}}$

C) $f(x) = 2\left(\frac{1}{8}\right)^{\frac{x}{3}}$

D) $f(x) = 2\left(\frac{1}{16}\right)^{\frac{x}{4}}$

13

The population of trees in a forest has been decreasing by 6 percent every 4 years. The population at the beginning of 2015 was estimated to be 14,000. If P represents the population of trees in the forest t years after 2015, which of the following equations best defines P ?

A) $P = 14{,}000(0.06)^{\frac{t}{4}}$

B) $P = 14{,}000 + 0.94(4t)$

C) $P = 14{,}000(0.94)^{4t}$

D) $P = 14{,}000(0.94)^{\frac{t}{4}}$

14

$$v = 40(1.6)^{t}$$

The given equation models the resale value of a collectible toy t years after its first release. Which of the following equations best models the resale value, v, of the toy m months after its first release?

A) $v = 40(1.6)^{m}$

B) $v = 40(1.6)^{\frac{m}{12}}$

C) $v = 40(1.6)^{12m}$

D) $v = 40(1.6)^{\frac{12}{m}}$

15

$$N(h) = 1{,}000(0.97)^{4h}$$

The function N models the number of bacteria in a petri dish after h hours. According to the model, the number of bacteria is predicted to decrease by 3% every k minutes. What is the value of k ?

A) $\frac{1}{4}$

B) 4

C) 15

D) 240

16

The amount of data, in gigabytes, stored in a database increases by 2% every 15 hours. If 16 gigabytes worth of data are currently stored in the database, which of the following functions g gives the amount of data, in gigabytes, that will be stored in the database t days from now?

A) $g(t) = 16(2)^{15t}$

B) $g(t) = 16(1.02)^{\frac{t}{15}}$

C) $g(t) = 16(1.02)^{\frac{5t}{8}}$

D) $g(t) = 16(1.02)^{\frac{8t}{5}}$

EXERCISE 2: Answers for this exercise start on page 345.

1

In 2004, a streaming service had 10,000 subscribers. The number of subscribers increased by 160% each year. Which of the following equations best models the number of subscribers, s, the streaming service had t years after 2004?

A) $s = 10{,}000 + 16{,}000t$
B) $s = 10{,}000(1.16)^t$
C) $s = 10{,}000(1.6)^t$
D) $s = 10{,}000(2.6)^t$

2

$$r(t) = 11(0.92)^t$$

The function r models the roundworm population, in thousands, in a pond after t years. Which of the following is the best interpretation of the number 0.92 in this context?

A) The roundworm population decreases by 0.92 thousand each year.
B) It will take 0.92 years for the roundworm population to halve.
C) The proportion of the roundworm population that remains after each year is 0.92.
D) The roundworm population decreases by 0.92 percent each year.

3

Each week, the weight of a certain crocodile increases by 0.75% of its initial birth weight. Which of the following functions best models the weight of this crocodile over time?

A) Decreasing exponential
B) Decreasing linear
C) Increasing exponential
D) Increasing linear

4

The function $f(x) = 2.9(1.07)^x$ gives the estimated value, in millions of dollars, of a missing piece of artwork, where x is the number of years since the artwork disappeared, and $0 \leq x \leq 6$. If $y = f(x)$ is graphed in the xy-plane, which of the following is the best interpretation of the y-intercept of the graph in this context?

A) The average estimated value of the artwork was 2.9 million dollars during the first 6 years of its disappearance.
B) The average estimated value of the artwork was 3.1 million dollars during the first 6 years of its disappearance.
C) The estimated value of the artwork was 2.9 million dollars at the time it disappeared.
D) The estimated value of the artwork was 3.1 million dollars at the time it disappeared.

5

$$C(t) = 100(2)^{\frac{t}{5}}$$

To examine how a certain virus spreads, scientists introduced the virus to cells in a test tube and found that the number of infected cells in the test tube grew exponentially over time. The given function C models the number of infected cells in the test tube t days after the virus was introduced. Which is the best interpretation of $(2)^{\frac{t}{5}}$ in this context?

A) The predicted number of infected cells in the test tube doubled every 5 days.
B) The predicted number of infected cells in the test tube grew by a factor of 5 every two days.
C) The predicted number of infected cells in the test tube doubled every day.
D) The predicted number of infected cells in the test tube grew by a factor of 5 every day.

6

Laura purchases a stock valued at \$326.50. According to her model, the stock is predicted to have a value of \$413.60 after 8 years. If Laura's model can be represented by the exponential function V, where $V(t)$ is the value of the stock, in dollars, t years after its purchase, which of the following equations could define V ?

A) $V(t) = 87.10(1.03)^t$
B) $V(t) = 87.10(1.3)^t$
C) $V(t) = 326.50(1.03)^t$
D) $V(t) = 326.50(1.3)^t$

7

For the function f, the value of $f(x)$ is equal to half the value of x. Which of the following is the best description of this function?

A) Decreasing linear
B) Decreasing exponential
C) Increasing linear
D) Increasing exponential

8

Ben paid off a loan by following a repayment plan. On the first day of the repayment plan, Ben paid \$30. On each day after the first, Ben paid 1.5 times the amount he had paid the previous day. If r represents the amount, in dollars, Ben paid on the nth day of the repayment plan, which equation gives r in terms of n ?

A) $r = 30(1.5)^n$
B) $r = 30(1.5)^{n-1}$
C) $r = 30(2.5)^n$
D) $r = 30(2.5)^{n-1}$

9

$$h(t) = 10{,}000(3)^{bt}$$

The given function h models the number of honeybees in a certain beehive, where b is a constant and t is the number of days after the beehive was first found. According to the model, there were 30,000 honeybees in the beehive 10 days after the beehive was first found. What is the value of b ?

10

$$W(t) = 830(1.09)^{\frac{6}{8}t}$$

The function W gives the value, in dollars, of a certain investment after t years. If the value of the investment increases by 9% every n months, what is the value of n ?

11

$$g(x) = 256(0.61)^{\frac{x}{30}}$$

The exponential function g is defined by the given equation. The equation can be rewritten as $g(x) = 256\left(1 - \frac{k}{100}\right)^{\frac{x}{10}}$, where k is a constant. What is the value of k, to the nearest whole number?

A) 15
B) 20
C) 27
D) 77

12

Since the beginning of 1990, the value of a certain diamond pendant has decreased by 4% from the beginning of each year to the beginning of the following year. At the beginning of 1991, the value of the pendant was \$840. Which of the following equations best models the value, in dollars, of the pendant t years after the beginning of 1990?

A) $v = 840(0.96)^t$

B) $v = 840(1.04)(0.96)^t$

C) $v = 875(0.96)^t$

D) $v = 875 - 875(0.04)^t$

13

The table below shows the price P, in dollars, of a barrel of crude oil t days after the beginning of an oil shortage.

Number of days, t	Price, P (dollars)
0	50.00
10	60.00
20	72.00
30	86.40
40	103.68

If the equation $P = 50(r)^{\frac{t}{n}}$ is used to model the relationship between t and P, which of the following could be the values of r and n ?

A) $r = 1.2$ and $n = 5$

B) $r = 1.44$ and $n = 5$

C) $r = 1.2$ and $n = 20$

D) $r = 1.44$ and $n = 20$

14

The function f is defined for all $x \geq 0$. Which of the following equivalent forms of the function f displays, as a constant or coefficient, the minimum value of f, where $x \geq 0$?

A) $f(x) = 50(2.5)(1.25)^{x-2}$

B) $f(x) = 125(0.8)(1.25)^{x-1}$

C) $f(x) = 64(1.25)^{x+1}$

D) $f(x) = 80(0.64)(1.25)^{x+2}$

15

$$C = 4(1.002)^{2t}$$

The given equation can be used to model the number of cars, in millions, registered in a certain state t years after 2009. According to the model, the number of cars registered in the state increases by n% every 6 months. What is the value of n ?

A) 0.002

B) 0.04

C) 0.2

D) 2

16

A music festival increases the number of tickets offered for sale each year by about 5% of the number offered for sale the previous year. In its fourth year, the music festival offered 40,000 tickets for sale. Which of the following equations best models the number of tickets offered for sale in the nth year of the music festival?

A) $T = 40{,}000(0.05)^{n-4}$

B) $T = 40{,}000(1.05)^{n}$

C) $T = 40{,}000(1.05)^{n-4}$

D) $T = 40{,}000(1.05)^{n+4}$

17

The function $p(t) = 870(1.14)^t$ models the number of thistle plants in a field from 2000 to 2020, where t is the number of years after 2000. Which of the following functions best models the number of thistle plants in the field from 2000 to 2020, where n is the number of years after <u>2003</u> ?

A) $q(n) = 870(1.14)^{n-3}$

B) $q(n) = 870(1.14)^{3n}$

C) $q(n) = 870(1.14)^3(1.14)^{\frac{n}{3}}$

D) $q(n) = 870(1.14)^3(1.14)^n$