5 Functions

A function is a machine that takes an input, transforms it, and spits out an output. In math, functions are typically denoted by $f(x)$, with x being the input. So for the function

$$f(x) = x^2 + 1$$

every input is squared and then added to one to get the output. It's important to understand that x is a completely arbitrary label—it's just a placeholder for the input. Anything can be an input, including other functions and expressions defined in terms of x:

$$f(2x) = (2x)^2 + 1$$
$$f(a) = a^2 + 1$$
$$f(b+1) = (b+1)^2 + 1$$
$$f(g(x)) = (g(x))^2 + 1$$
$$f(\text{Panda}) = (\text{Panda})^2 + 1$$

Note the careful use of parentheses. In the first equation, for example, writing $2x^2$ instead of $(2x)^2$ would be incorrect. Wrap the input in parentheses and you'll never go wrong.

EXAMPLE 1: The function f is defined by $f(x) = (x+1)^x$. What is the value of $f(0) + f(1) + f(2) + f(3)$?

Just plug in the given inputs.

$$\begin{aligned} f(0) + f(1) + f(2) + f(3) &= (0+1)^0 + (1+1)^1 + (2+1)^2 + (3+1)^3 \\ &= 1^0 + 2^1 + 3^2 + 4^3 \\ &= 1 + 2 + 9 + 64 \\ &= \boxed{76} \end{aligned}$$

EXAMPLE 2: The function g is defined by $g(x) = 7x + 8$. For what value of x is $g(x) = -13$?

Here, we're asked to find the input that results in an output of -13. To find this input, we can set up an equation:

$$\begin{aligned} g(x) &= -13 \\ 7x + 8 &= -13 \\ 7x &= -21 \\ x &= \boxed{-3} \end{aligned}$$

EXAMPLE 3: The functions f and g are defined by $f(x) = 3x - 2$ and $g(x) = 2x - 18$. If the function h is defined by $h(x) = f(x) + g(x)$, what is the value of $h(9)$?

Functions can be added, subtracted, multiplied, and divided to create new functions. Here, function h is created by adding functions f and g:

$$h(x) = f(x) + g(x) = (3x - 2) + (2x - 18) = 5x - 20$$

Therefore, $h(9) = 5(9) - 20 = \boxed{25}$.

EXAMPLE 4: Functions f and g are defined by $f(x - 1) = 6x$ and $g(x) = x + 3$. What is the value of $f(g(2))$?

Whenever you see composite functions (functions that are written inside of another function), start from the inside and work your way out. First,

$$g(2) = 2 + 3 = 5$$

Now we have to figure out the value of $f(5)$.

Well, we can plug in $x = 6$ into $f(x - 1)$ to get $f(5) = 6(6) = \boxed{36}$. Don't be thrown off by the $x - 1$ in $f(x - 1)$. It simply means that $x - 1$ is the input rather than x. Remember that the input can be any expression.

EXAMPLE 5: Functions f and g are defined by $f(x) = x + 1$ and $g(x) = \frac{x}{2}$. If $f(g(k)) = 10$, what is the value of k ?

Again, we start from the inside and work our way out. Since $g(k) = \frac{k}{2}$,

$$f(g(k)) = f\left(\frac{k}{2}\right) = \frac{k}{2} + 1$$

Setting this expression equal to 10, we get

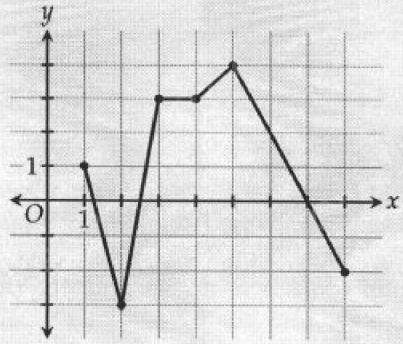
$$\begin{aligned} \frac{k}{2} + 1 &= 10 \\ \frac{k}{2} &= 9 \\ k &= \boxed{18} \end{aligned}$$

As we've mentioned, a function takes an input and returns an output. Well, these input and output pairs allow us to graph any function as a set of points in the xy-plane, with the input as x and the output as y. Questions on the SAT will often use phrases like "the graph of $y = f(x)$" or "the graph of $y = g(x)$" to make it clear that the output of a function is being represented as the y-coordinate. In other words, a function like $f(x) = x^2 + 1$ can be viewed as $y = x^2 + 1$.

Essentially, in a graphing context, $f(x)$ and y are interchangeable. So, a statement that $f(x) > 0$, for example, means that all the points on the graph have positive y-values (i.e. the graph is always above the x-axis).

Learning to think of points on a graph as the inputs and outputs of a function will help you solve many SAT questions.

EXAMPLE 6:



The graph of $y = f(x)$ is shown in the xy-plane. For what value of x does $f(x)$ reach its maximum?

Again, when it comes to graphs, think of $f(x)$ as the y. So we're looking for the point on the graph with the highest y-value, the "peak" of the graph. That point is $(5, 4)$, so the value of x we're looking for is $\boxed{5}$.

EXAMPLE 7: The function f is defined by $f(x) = ax^2 + 3$, where a is a constant. The graph of $y = f(x)$ in the xy-plane passes through the point $(1, 2)$. What is the value of a ?

Remember—a point is just an input and an output, an x and a y. Because $(1, 2)$ is a point on the graph of the function, we can plug in 1 for x and 2 for y. Doing so gives us an equation to solve for a:

$$2 = a(1)^2 + 3$$
$$2 = a + 3$$
$$a = \boxed{-1}$$

EXAMPLE 8: The function f is defined by $f(x) = 3(6)^x - 8$. What is the y-intercept of the graph of $y = f(x)$ in the xy-plane?

A) $(0, -8)$ B) $(0, -5)$ C) $(0, 6)$ D) $(0, 10)$

The y-intercept is the point at which the graph crosses the y-axis. The y-intercept always has an x-coordinate of 0, so to find the y-coordinate, we can just plug $x = 0$ into the function:

$$f(0) = 3(6)^0 - 8 = 3(1) - 8 = -5$$

Therefore, the y-intercept of the graph is $(0, -5)$. Answer $\boxed{(B)}$.

EXAMPLE 9: The function g is defined by $g(x) = 8x + 4$. What is the x-intercept of the graph of $y = g(x)$ in the xy-plane?

A) $(-8, 0)$ B) $(-2, 0)$ C) $\left(-\frac{1}{2}, 0\right)$ D) $(4, 0)$

An x-intercept is a point at which the graph crosses the x-axis. By definition, x-intercepts have a y-coordinate of 0, so to find the x-coordinate, we can set $g(x) = 0$ and solve for x:

$$\begin{aligned} 8x + 4 &= 0 \\ 8x &= -4 \\ x &= -\frac{1}{2} \end{aligned}$$

In this case, we just have one x-intercept at $\left(-\frac{1}{2}, 0\right)$. Answer $\boxed{(C)}$.

EXAMPLE 10:

$$q(x) = x(x + 5)(x - 6)$$

If the given function q is graphed in the xy-plane, where $y = q(x)$, what is an x-intercept of the graph?

A) $(-6, 0)$ B) $(-5, 0)$ C) $(-1, 0)$ D) $(5, 0)$

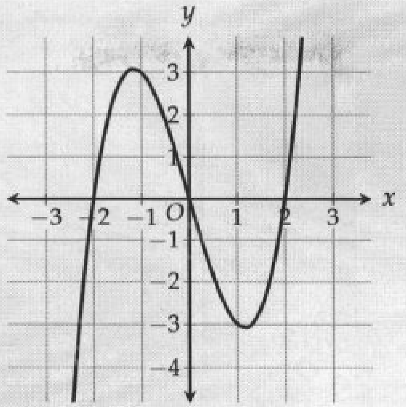
To find the x-intercepts, we set $q(x) = 0$ and solve for x:

$$x(x + 5)(x - 6) = 0$$

The product on the left side of the equation is equal to 0 only when $x = 0$, $x + 5 = 0$, or $x - 6 = 0$. The solutions are then $x = 0, -5$, and 6, which means the x-intercepts are at $(0, 0)$, $(-5, 0)$, and $(6, 0)$. Of these, only $(-5, 0)$ is in the answer choices. Answer $\boxed{(B)}$.

Note that x, $x + 5$, and $x - 6$ are **factors** of $q(x)$. These factors make it easy to identify the x-intercepts.

EXAMPLE 11:



The graph of $y = h(x)$ is shown, where the function h is defined by $h(x) = ax^3 + bx$, where a and b are constants. For how many values of x does $h(x) = 0$?

We can reword this question in several ways: How many different inputs result in an output of 0? How many points on the graph have a y-value of 0?

Essentially, this question is just asking for the number of x-intercepts. Since the given graph crosses the x-axis 3 distinct times, there are $\boxed{3}$ values of x that result in $h(x) = 0$.

Note that the x-intercepts of the graph are $(-2, 0)$, $(0, 0)$, and $(2, 0)$. The x-values of $-2, 0$, and 2 indicate that $(x + 2), x$, and $(x - 2)$, respectively, must be factors of $h(x)$. With these factors, the equation of $h(x)$ should look something like

$$h(x) = kx(x - 2)(x + 2)$$

where k is a constant.

The **zeros, roots, and x-intercepts** of a function are all just different terms for the same thing—the values of x that make $f(x) = 0$ (or $y = 0$). Graphically, they refer to the values of x where the function crosses the x-axis.

If the graph of a function has an x-intercept at $(n, 0)$, then $(x - n)$ must be a factor of that function.

EXAMPLE 12:

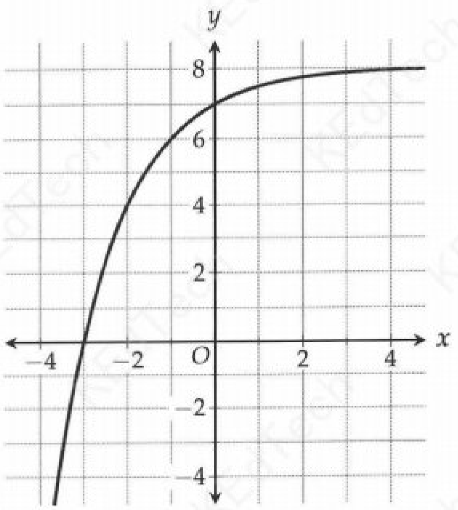
$$f(x) = \frac{4}{(x - 5)^2}$$

The function f is given. For what value of x is $f(x)$ undefined?

A function is undefined for any input that results in an operation that is "not allowed" or gives an undefined answer. Division by zero is the most common case. Since $f(x)$ has a denominator of zero when $x = 5$, $f(x)$ is undefined when $x = \boxed{5}$.

EXERCISE 1: Answers for this exercise start on page 329.

1



What is the y-intercept of the graph shown?

A) $(-3,0)$
B) $(7,0)$
C) $(0,-3)$
D) $(0,7)$

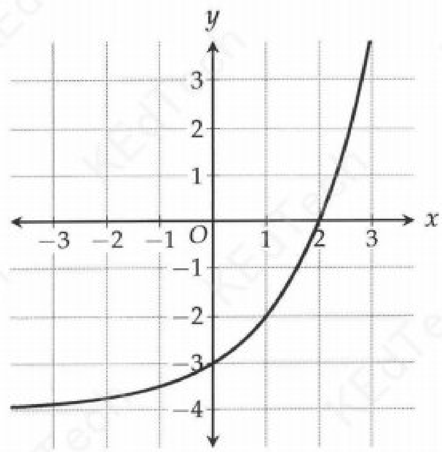
2

x	$f(x)$
0	20
1	21
3	29

For the function f, the table shows three values of x and their corresponding values of $f(x)$. Which equation could define f ?

A) $f(x) = 20x$
B) $f(x) = x + 20$
C) $f(x) = x - 20$
D) $f(x) = x^2 + 20$

3



The graph of the exponential function f is shown in the xy-plane. For what value of x does $f(x) = 0$?

A) -4
B) -3
C) 0
D) 2

4

What is the y-coordinate of the y-intercept of the graph of $y = -4\left(\frac{1}{2}\right)^x + 9$?

5

The function f is defined by $f(x) = \dfrac{16 + x^2}{2x}$. What is the value of $f(-4)$?

A) -8
B) -4
C) 4
D) 8

6

A ball falls from a platform that is 150 feet above the ground. The function $h(t) = 150 - 16t^2$ gives the ball's height, in feet, above the ground t seconds after it falls. Which of the following is the best interpretation of $h(3) = 6$ in this context?

A) The ball is 3 feet above the ground 6 seconds after it falls.

B) The ball is 3 feet below the platform 6 seconds after it falls.

C) The ball is 6 feet above the ground 3 seconds after it falls.

D) The ball is 6 feet below the platform 3 seconds after it falls.

7

If $f(x) = x^2$, for which of the following values of c is $f(c) < c$?

A) $\dfrac{1}{2}$

B) 1

C) $\dfrac{3}{2}$

D) 2

8

$$f(x) = 4x - 3$$
$$g(x) = 3x + 5$$

The functions f and g are defined by the given equations. Which of the following is equal to $f(8)$?

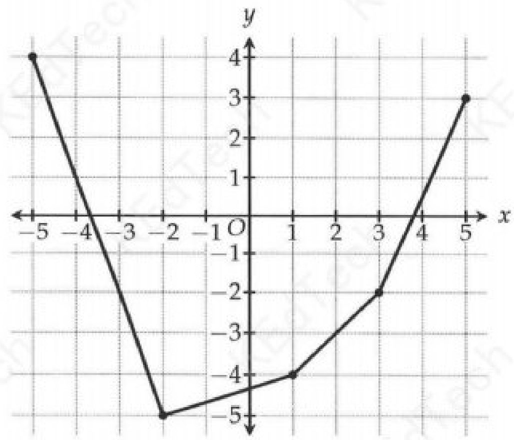
A) $g(1)$

B) $g(3)$

C) $g(5)$

D) $g(8)$

9



The graph of $y = f(x)$ is shown in the xy-plane. If $f(a) = f(3)$, which of the following could be the value of a ?

A) -4

B) -3

C) -2

D) 1

10

$$f(x) = x^2 + 1$$
$$g(x) = x^2 - 1$$

The functions f and g are defined by the given equations. What is the value of $f(g(2))$?

A) 3

B) 5

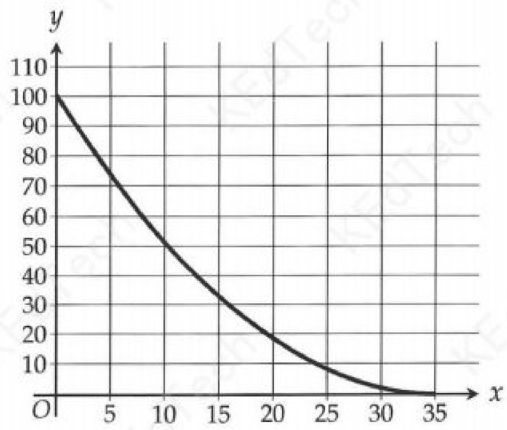
C) 10

D) 17

11

The function f is defined by $f(x) = 3x + 2$ and the function g is defined by $g(x) = f(2x) - 1$. What is the value of $g(10)$?

12



The graph gives the atmospheric pressure y, in kilopascals, that corresponds to an elevation of x kilometers above sea level. Which statement is the best interpretation of the y-intercept in this context?

A) The atmospheric pressure at sea level is about 35 kilopascals.

B) The atmospheric pressure at sea level is about 100 kilopascals.

C) The atmospheric pressure at 35 kilometers above sea level is about 0 kilopascals.

D) The atmospheric pressure at 100 kilometers above sea level is about 0 kilopascals.

13

The function f is defined by $f(x) = -3x + 5$, and $\frac{1}{2}f(a) = 10$, where a is a constant. What is the value of a ?

A) -8

B) -5

C) 5

D) 8

14

The graph of $y = f(x)$ in the xy-plane has exactly two x-intercepts at $(r, 0)$ and $(s, 0)$, where r and s are positive constants. Which of the following equations could define f ?

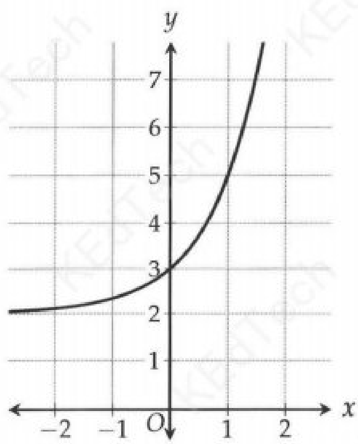
A) $f(x) = (x + r)(x + s)$

B) $f(x) = x(x + r)(x + s)$

C) $f(x) = (x - r)(x - s)$

D) $f(x) = x(x - r)(x - s)$

15



The graph of $y = 3^x + b$ is shown, where b is a constant. What is the value of b ?

A) 1

B) 2

C) 3

D) 4

16

Function f is defined by $f(x) = x(x+2)(x+6)$. Which table of values represents $y = f(x-2)$?

A)

x	y
-8	0
-4	0
-2	0

B)

x	y
-4	0
0	0
2	0

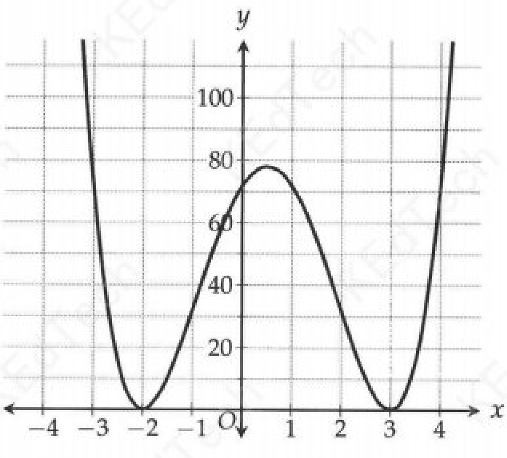
C)

x	y
-2	0
0	0
4	0

D)

x	y
2	0
4	0
8	0

17



Which of the following could be the equation of the graph shown?

A) $y = (x+2)^2(x-3)^2$

B) $y = (x-2)^2(x+3)^2$

C) $y = 2(x+2)^2(x-3)^2$

D) $y = 2(x-2)^2(x+3)^2$

18

Function f is defined by $f(x) = (x+2)(x-2)^2$. The graph of $y = f(x)$ in the xy-plane passes through the point (p, q), where $-2 \leq p \leq 2$. Which of the following is NOT a possible value of q ?

A) 12

B) 8

C) 4

D) 0

19

If $f(x+1) = \frac{1}{4}f(x)$, which of the following equations could define the function f ?

A) $f(x) = \frac{1}{4}\left(\frac{1}{8}\right)^x$

B) $f(x) = 8\left(\frac{1}{4}\right)^x$

C) $f(x) = 8(4)^x$

D) $f(x) = \frac{1}{4}(8)^x$

20

x	0	1	2
$f(x)$	-2	3	18

Function f is defined by $f(x) = ax^2 + b$, where a and b are constants. The table gives three values of x and their corresponding values of $f(x)$. What is the value of $f(3)$?

A) 23

B) 39

C) 43

D) 56

EXERCISE 2: Answers for this exercise start on page 330.

1

The function $g(x) = 0.002x^2 + 5x$ models the revenue, in dollars, generated by an online advertisement that is clicked half as many times as it is viewed, where x is the number of times it is viewed. Which of the following is the best interpretation of $g(2{,}000) = 18{,}000$ in this context?

A) If the advertisement is clicked 2,000 times, then it will generate 18,000 dollars in revenue.

B) If the advertisement is viewed 2,000 times, then it will generate 18,000 dollars in revenue.

C) If the advertisement is viewed 18,000 times, then it will generate 2,000 dollars in revenue.

D) If the advertisement is clicked 18,000 times, then it will generate 2,000 dollars in revenue.

2

The function f is defined by $f(x) = \dfrac{x+18}{x-6}$. For what value of x does $f(x) = -1$?

3

Function g is defined by $g(x) = \sqrt{3x}$. If $g(a) = 6$, what is the value of a ?

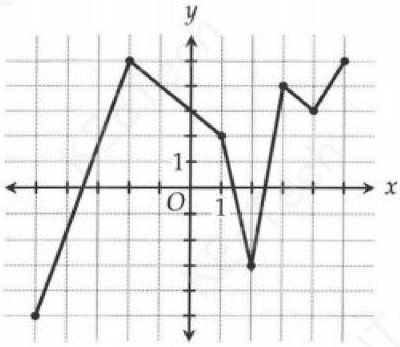
A) 3

B) 6

C) 9

D) 12

4



The graph of $y = f(x)$ is shown in the xy-plane. For how many values of x does $f(x) = 3$?

A) Two

B) Three

C) Four

D) Five

5

The function f is defined by $f(x) = \dfrac{a^{-x}}{b}$, where a and b are positive constants. What is the y-intercept of the graph of $y = f(x)$ in the xy-plane?

A) $\left(0, \dfrac{1}{b}\right)$

B) $\left(0, \dfrac{a}{b}\right)$

C) $\left(0, \dfrac{1}{ab}\right)$

D) $(0, 1)$

6

x	$f(x)$	$g(x)$
-2	3	4
-1	5	2
0	-2	-3
1	3	5
2	6	7
3	7	1

For the functions f and g, the table gives six values of x and their corresponding values of $f(x)$ and $g(x)$. If $g(c) = 5$, where c is a constant, what is the value of $f(c)$?

A) -2
B) 3
C) 5
D) 6

7

$$f(x) = -x(x-a)(x+a)(x+b)(x+b)$$

In the given function f, a and b are positive constants and $a \neq b$. For how many distinct values of x does $f(x) = 0$?

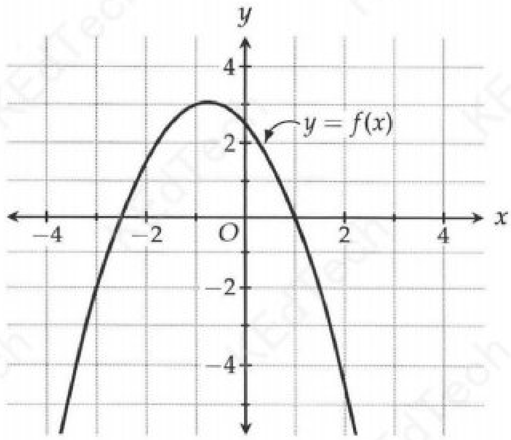
A) Two
B) Three
C) Four
D) Five

8

In the xy-plane, the graph of $y = f(x)$ reaches its maximum value at the point $(3, f(3))$. The function g is defined by $g(x) = f(x) + 7$. At which of the following points in the xy-plane does the graph of $y = g(x)$ reach its maximum value?

A) $(10, f(10) + 7)$
B) $(f(3), f(3) + 7)$
C) $(3, f(10))$
D) $(3, f(3) + 7)$

9



The graph of $y = f(x)$ is shown in the xy-plane. If the function g is defined by $g(x) = (x+3)(x-1)$, for which of the following values of x is $f(x) > g(x)$?

A) -3
B) -2
C) 1
D) 2

10

x	$f(x)$
-4	3
-2	5
0	2
2	16
3	4
4	8

For the function f, the table shows several values of x and their corresponding values of $f(x)$. Function g is defined by $g(x) = 2f(x)$. What is the value of k if $g(k) = 8$?

A) 2
B) 3
C) 4
D) 8

11

The function f is defined by $f(x) = \dfrac{(x+3)(12-x)}{x}$. The value of $f(a-9)$ is 0, where a is a constant. What is the greatest possible value of a ?

12

Function g is defined by $g(x) = \dfrac{x-8}{2}$. The graph of $y = g(x)$ in the xy-plane passes through the points $(20, c)$ and (c, d), where c and d are constants. What is the value of d ?

A) -5
B) -1
C) 3
D) 6

13

The graph of $y = f(x)$ has a y-intercept at $(0, 12)$ and x-intercepts at $(-3, 0)$ and $(2, 0)$. Which of the following equations could define f ?

A) $f(x) = (x+3)^2(x-2)$
B) $f(x) = (x+3)(x-2)^2$
C) $f(x) = (x-3)^2(x+2)$
D) $f(x) = (x-3)(x+2)^2$

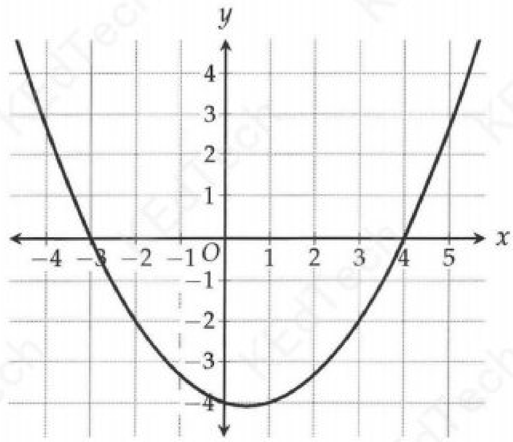
14

$$f(x) = \sqrt{x-2}$$

The function f is defined by the given equation. Which of the following is equal to $f(18) - f(11)$?

A) $f(3)$
B) $f(5)$
C) $f(6)$
D) $f(7)$

15



An equation for the graph shown is $y = a(x+3)(x+b)$, where a and b are constants. What is the value of ab ?

A) -2
B) $-\dfrac{4}{3}$
C) $\dfrac{1}{2}$
D) 8

16

The function f is defined by $f(x) = \dfrac{11}{3}x + 34$, and the function g is defined by $g(x) = \dfrac{5}{3}x - 8$. If the function h is defined by $h(x) = f(x) - g(x)$, what is the x-coordinate of the x-intercept of the graph of $y = h(x)$ in the xy-plane?

17

The function f is defined by $f(x) = 125 \cdot a^x - b$, where a and b are constants. In the xy-plane, the graph of $y = f(x)$ has an x-intercept at $(3, 0)$ and a y-intercept at $(0, b - 3)$. What is the value of b ?

18

The function f is defined by $f(x) = \dfrac{x + 32}{x + b}$, where b is a constant. If $f(-8) = 4$, what is the value of $f(10)$?

19

The function f is defined by $f(x) = 7(4)^{-x} + 13$. Which of the following statements must be true?

I. $f(x) > 7$
II. $f(x) \leq 20$

A) I only
B) II only
C) Neither I nor II
D) I and II

20

The function f is defined by $f(x) = b \cdot a^x$, where a and b are positive constants, and $g(x) = f(x + 2)$. The graph of $y = f(x)$ in the xy-plane passes through the points $(0, 36)$ and $\left(-3, \dfrac{9}{16}\right)$. Which equation defines the function g ?

A) $g(x) = 216 \cdot 3^x$
B) $g(x) = 324 \cdot 3^x$
C) $g(x) = 288 \cdot 4^x$
D) $g(x) = 576 \cdot 4^x$

21

Function f is defined by $f(x) = a\sqrt{x} + b$, where a and b are constants. In the xy-plane, the graph of $y = f(x)$ has an x-intercept at $(16, 0)$, and the graph of $y = f(x) - 10$ has an x-intercept at $(9, 0)$. What is the value of b ?

22

$$f(x) = b\sqrt{a - x} + 7\sqrt{a - x}$$

For the function f, a and b are constants and $f(16) = 0$. If $f(9) > 0$, which of the following could be the value of $a + b$?

A) -3
B) 7
C) 9
D) 15