3

Manipulating & Solving Equations

On the SAT, there is a huge emphasis on equations. To get these types of questions right, you must learn how to isolate the variables and expressions on one side of the equation. Let's dive into some tips and techniques.

1. Don't forget to combine like terms

You should be ruthless in finding like terms and combining them. Doing so will simplify things and allow you to figure out the next step.

EXAMPLE 1: If $2(a+b+2c+3d+1) = 3a+2b+4c+6d$, what is the value of a ?

The same four variables are on both sides of the equation. So before we say to ourselves that this question is impossible because there are 4 variables and one equation, we should first get all the terms on one side and combine the ones that are alike. Sounds simple but you won't believe how many students judge a problem before they've taken the first step.

$$2(a+b+2c+3d+1) = 3a+2b+4c+6d$$
$$2a+\cancel{2b}+\cancel{4c}+\cancel{6d}+2 = 3a+\cancel{2b}+\cancel{4c}+\cancel{6d}$$
$$\boxed{2} = a$$

As you can see, the $b, c,$ and d terms canceled quite nicely.

2. Square equations correctly

When squaring equations to remove a square root, the most important thing to remember is that you're not squaring individual elements—you're squaring the entire side.

EXAMPLE 2:

$$\sqrt{ab} = a - b$$

If $a > 0$ and $b > 0$, the given equation is equivalent to which of the following?

A) $ab = a^2 - b^2$ B) $ab = a^2 + b^2$ C) $2ab = a^2 - b^2$ D) $3ab = a^2 + b^2$

The square root in the problem should scream to you that the equation should be squared. Most students know they have to get rid of the square root, but all too often, their approach is to square each individual element:

$$ab = a^2 - b^2$$

This result, of course, is incorrect. When modifying equations, you must apply any given operation to the entire SIDE, like so:

$$(\sqrt{ab})^2 = (a - b)^2$$

If it helps, wrap each side in parentheses before applying the operation. By the way, the same holds true for all other operations, including multiplication and division. When you multiply or divide both sides of an equation, what you're actually doing is wrapping each side in parentheses, but because of the distributive property, it just so happens that multiplying or dividing each individual element gets you the same result. For example, if we had the equation

$$x + 2 = y$$

and we wanted to multiply both sides by 3, what we're actually doing is

$$3(x + 2) = 3(y)$$

which turns out to be the same as

$$3x + 6 = 3y$$

Anyway, back to the problem:

$$(\sqrt{ab})^2 = (a - b)^2$$
$$ab = a^2 - 2ab + b^2$$
$$3ab = a^2 + b^2$$

The answer is $\boxed{(D)}$.

EXAMPLE 3:

$$\sqrt{x+3}+y=z$$

The given equation relates the positive real numbers $x, y,$ and z. Which equation correctly expresses x in terms of y and z ?

A) $x=z^2-y^2-3$ B) $x=z^2+y^2-3$ C) $x=(z-y)^2-3$ D) $x=(z+y)^2-3$

Another common mistake is squaring each side before the equation is "ready" to be squared. For this example, here's what we'd get if we squared each side right away:

$$(\sqrt{x+3}+y)^2=z^2$$

Imagine expanding this out and trying to isolate x. We'd be in a world of pain. The right approach is to isolate the square root on the left by subtracting y from both sides:

$$\sqrt{x+3}=z-y$$

Now we can square each side:

$$\begin{aligned} x+3 &= (z-y)^2 \\ x &= (z-y)^2-3 \end{aligned}$$

Answer $\boxed{\text{(C)}}$.

3. Square root equations correctly

Now, when it comes to taking the square root of both sides, most students forget the plus or minus ($\pm$). Always remember that an equation such as $x^2=25$ has two solutions:

$$\begin{aligned} \sqrt{x^2} &= \sqrt{25} \\ x &= \pm 5 \end{aligned}$$

However, the plus or minus only applies when you're taking the square root to solve an equation. Otherwise, square roots, by definition, always refer to the positive root. That's why $\sqrt{9}=3$, not ± 3, and $\sqrt{x}=-3$ has no real solution. It's only in situations where the square root is used to solve an equation that the plus or minus gets tacked on.

EXAMPLE 4: If $(x+3)^2=121$, what is the sum of the two possible values of x ?

$$\begin{aligned} (x+3)^2 &= 121 \\ \sqrt{(x+3)^2} &= \pm\sqrt{121} \\ x+3 &= \pm 11 \\ x &= -3 \pm 11 \end{aligned}$$

So x could be either 8 or -14. The sum of those two possibilities is $\boxed{-6}$.

4. Cross-multiply when fractions are set equal to each other

Whenever a fraction is equal to another fraction,

$$\frac{a}{b} = \frac{c}{d}$$

you can cross-multiply: $ad = bc$.

EXAMPLE 5: If $\frac{4}{5}x = \frac{10}{3}$, what is the value of x ?

$$\begin{aligned} \frac{4}{5}x &= \frac{10}{3} \\ 12x &= 50 \\ x &= \boxed{\frac{25}{6}} \end{aligned}$$

EXAMPLE 6:

$$\frac{2}{b} - \frac{5c}{a+bc} = 0$$

The given equation relates the positive numbers a, b, and c. Which equation correctly expresses a in terms of b and c ?

A) $2bc$ B) $\frac{3}{2}bc$ C) $\frac{2}{5bc} - bc$ D) $\frac{2-5c}{b-bc}$

$$\begin{aligned} \frac{2}{b} - \frac{5c}{a+bc} &= 0 \\ \frac{2}{b} &= \frac{5c}{a+bc} \\ 2(a+bc) &= 5bc \\ 2a + 2bc &= 5bc \\ 2a &= 3bc \\ a &= \frac{3}{2}bc \end{aligned}$$

Answer $\boxed{(B)}$.

5. Factor to isolate variables that are common to multiple terms

Some equations have variables that are tougher to isolate. For a lot of these equations, you will have to do some shifting around to factor out the variable you want.

EXAMPLE 7:

$$b = \frac{a}{3a + c}$$

Which of the following expresses a in terms of b and c ?

A) $\dfrac{bc}{1 - 3b}$ B) $\dfrac{bc}{3b + 1}$ C) $\dfrac{1 - 3b}{bc}$ D) $\dfrac{3b + 1}{bc}$

$$\begin{aligned} b &= \frac{a}{3a + c} \\ b(3a + c) &= a \\ 3ab + bc &= a \\ bc &= a - 3ab \\ bc &= a(1 - 3b) \\ \frac{bc}{1 - 3b} &= a \end{aligned}$$

See what we did? We expanded everything out and put every term containing a on the right side. Then we were able to factor out a and isolate it. The answer is $\boxed{(A)}$.

6. Factor to get a product of expressions

EXAMPLE 8:

$$x^2(x - 3) - (x - 3) = 0$$

What is the smallest solution to the given equation?

We can factor out $(x - 3)$ to get

$$(x - 3)(x^2 - 1) = 0$$

If you didn't quite catch that, here's another way to see it. If we let $A = (x - 3)$, the equation becomes $x^2A - A = 0$, and we can factor out A to get $A(x^2 - 1) = 0$. Substituting $(x - 3)$ back in for A, we get $(x - 3)(x^2 - 1) = 0$. The takeaway here is that any shared term or expression can be factored out.

Looking at the result, we see that we have a difference of two squares, $x^2 - 1$, which can be factored further:

$$\begin{aligned} (x - 3)(x^2 - 1) &= 0 \\ (x - 3)(x + 1)(x - 1) &= 0 \end{aligned}$$

With this product of expressions, it's easy to see that we have 3 possible cases (solutions): $x - 3$ is equal to 0, $x + 1$ is equal to 0, or $x - 1$ is equal to 0. So the solutions are $x = 3, -1,$ or 1. Of these, $\boxed{-1}$ is the smallest.

7. Be comfortable solving for expressions, rather than any one variable

EXAMPLE 9: If $3x + 9y = 9$, what is the value of $x + 3y$?

Get in the habit of looking for what you want **before** you solve for anything specific. Is there any way to get the answer without solving for x and y?

Yes! Dividing both sides of the given equation by 3 gives $x + 3y = \boxed{3}$.

EXAMPLE 10: If $4\left(\dfrac{3-x}{2}\right) - 5 = 23$, what is the value of $\dfrac{3-x}{2}$?

The expression $\dfrac{3-x}{2}$ shows up in the equation itself, so we can solve for it directly without worrying about the value of x.

$$4\left(\frac{3-x}{2}\right) - 5 = 23$$

$$4\left(\frac{3-x}{2}\right) = 28$$

$$\left(\frac{3-x}{2}\right) = \boxed{7}$$

EXAMPLE 11: If $\dfrac{x}{y} = 3$, what is the value of $\dfrac{y}{2x}$?

A) $\dfrac{1}{6}$ B) $\dfrac{1}{3}$ C) $\dfrac{2}{3}$ D) $\dfrac{3}{2}$

Here, we have no choice but to solve for the expression. We're given x over y but we want y over x. We can flip the given equation to get

$$\frac{y}{x} = \frac{1}{3}$$

Then we can divide both sides by 2 to obtain the $\dfrac{y}{2x}$ we're looking for: $\dfrac{y}{2x} = \dfrac{1}{2 \cdot 3} = \dfrac{1}{6}$. Answer $\boxed{(A)}$.

8. Watch out for extraneous solutions and missing solutions

Extraneous solutions are values we may think are solutions but actually aren't. They are also called *false* solutions.

EXAMPLE 12:

$$\frac{(x-5)(x-3)}{x-5} = 0$$

What is a possible solution to the given equation?

A good first step to solving this equation might be to multiply both sides by $x - 5$ to get rid of the fraction. We would end up with

$$(x-5)(x-3) = 0$$

At this point, it's easy to see that either $x - 5 = 0$ or $x - 3 = 0$, so the possible solutions are $x = 5$ and $x = 3$.

However, plugging in $x = 5$ into the original equation results in $\frac{0}{0}$, which is undefined because of division by zero. Therefore, $x = 5$ is not a valid solution to this equation. It's an extraneous solution. The only possible solution here is $\boxed{3}$.

Extraneous solutions can also occur when working with square roots. As a simple example, the equation

$$\sqrt{x} = -4$$

has no solution because the square root of something can never be negative. However, if we square both sides, we get $x = (-4)^2$, which implies that $x = 16$ is a solution even though it clearly does not satisfy the original equation.

Why do these extraneous solutions occur? Well, manipulating equations in certain ways can bring about solutions that satisfy the transformed equation but not the original. Squaring both sides of an equation, for instance, has the effect of turning negative values into positive ones, and once this distinction is lost, false solutions can arise.

Let's take a look at an example of a missing solution.

EXAMPLE 13:

$$(x + 4)(x - 9) = (x + 4)$$

What is the sum of the solutions to the given equation?

Because $(x + 4)$ appears on both sides of the equation, let's simplify by dividing both sides by $(x + 4)$:

$$\frac{\cancel{(x+4)}(x-9)}{\cancel{(x+4)}} = \frac{\cancel{(x+4)}}{\cancel{(x+4)}}$$

$$(x - 9) = 1$$

$$x = 10$$

Plugging in $x = 10$ yields 14 on both sides of the original equation, so $x = 10$ is definitely a solution. But is 10 the only solution? There doesn't seem to be any more even though the question suggests that there's more than one.

The problem is that in dividing both sides by $(x + 4)$ to find one solution, we made another (more obvious) solution disappear: $x = -4$. Plugging in $x = -4$ into the original equation yields 0 on both sides, so it's clearly a solution. How come it was so easy to miss?

Remember that in math, division by zero is not allowed. So when we divided both sides by $(x + 4)$, we made the implicit assumption that x is not -4 (otherwise, $x + 4$ would be equal to zero). That division and all the steps we took afterwards were based on that assumption. However, in making that assumption, we neglected to consider that $x = -4$ itself might also be a solution. That's why it was missed. Since it is a solution, the answer here is $10 + (-4) = \boxed{6}$.

So what are the takeaways from this section?

- Dividing by an expression can cause you to miss obvious solutions.
- Multiplying by an expression can give rise to extraneous solutions.
- Equations with square roots and/or fractional expressions are more prone to have extraneous solutions. For these equations, get in the habit of plugging your results back into the original equation to see whether they are valid.

EXERCISE 1: Answers for this exercise start on page 319.

1

$$48 = \frac{8}{-x}$$

What is the solution to the given equation?

2

If $18 + 9x = 54$, what is the value of $2 + x$?

3

$$9 = -24 + 3x$$

Which equation has the same solution as the given equation?

A) $-3x = -15$
B) $3x = -15$
C) $-3x = -33$
D) $3x = -33$

4

If $3x - 8 = -23$, what is the value of $6x - 7$?

A) -5
B) -21
C) -30
D) -37

5

$$3(5 - 2x)(3 + x) = 0$$

What positive value of x is a solution to the given equation?

6

$$-3(r - 9) + 5(r - 9) = -4$$

What value of r is the solution to the given equation?

7

If $x^2 + 3x + 16 = 0$, what is the value of $5x^2 + 15x$?

8

If $\frac{12 - x}{9} = \frac{12 - x}{4}$, the value of $12 - x$ is between which of the following pairs of values?

A) -14 and -9
B) -8 and -4
C) -3 and 4
D) 5 and 13

9

$$\frac{64x^2}{36} = 64$$

What is the negative solution to the given equation?

10

If $\frac{4}{9} = \frac{8}{3}m$, what is the value of m ?

A) $\frac{1}{6}$

B) $\frac{2}{3}$

C) $\frac{5}{6}$

D) 6

11

$$y + 2kx = kx^2 + 5$$

In the given equation, k is a constant. If $y = 23$ when $x = 3$, what is the value of k ?

A) -6

B) 3

C) 6

D) 9

12

$$f = p\left(\frac{(1+j)^n - 1}{j}\right)$$

The given equation relates the positive numbers f, j, n, and p. Which of the following correctly expresses p in terms of f, j, and n ?

A) $p = \frac{fj}{(1+j)^n - 1}$

B) $p = \frac{(1+j)^n - 1}{fj}$

C) $p = \frac{f-j}{(1+j)^n - 1}$

D) $p = fj + 1 - (1+j)^n$

13

If $\frac{m}{2n} = 2$, what is the value of $\frac{n}{2m}$?

A) $\frac{1}{8}$

B) $\frac{1}{4}$

C) $\frac{1}{2}$

D) 1

14

$$-4d = 3e - h$$

The given equation relates the numbers d, e, and h. Which equation correctly expresses d in terms of e and h ?

A) $d = \frac{-3e-h}{4}$

B) $d = -\frac{3e}{4} - h$

C) $d = 3e + \frac{h}{4}$

D) $d = \frac{h-3e}{4}$

15

If $\frac{x}{6} = \frac{x+12}{42}$, what is the value of $\frac{6}{x}$?

A) $\frac{1}{3}$

B) 2

C) 3

D) 6

16

$$(n-4)^2 = (n+4)^2$$

What value of n satisfies the given equation?

17

$$\sqrt{x^2 - 25} = n$$

In the given equation, n is a positive constant. Which of the following is one of the solutions to the given equation?

A) $n + 5$
B) $\sqrt{n + 25}$
C) $\sqrt{n^2 + 5}$
D) $-\sqrt{n^2 + 25}$

18

If $3(x - 2y) - 3z = 0$, which of the following equations correctly expresses x in terms of y and z ?

A) $\dfrac{2y + 3z}{3}$
B) $2y + z$
C) $y + 2z$
D) $6y + 3z$

19

$$d = a\left(\frac{c+1}{24}\right)$$

Doctors use Cowling's rule, defined by the equation shown, to determine the correct dosage d, in milligrams, of medication for a child based on the adult dosage a, in milligrams, and the child's age c, in years. Ben is a patient who is in need of a certain medication. If a doctor uses Cowling's rule to prescribe Ben a dosage that is half the adult dosage, what is Ben's age, in years?

A) 7
B) 9
C) 11
D) 13

20

$$\frac{x^2 - 9}{x - 3} = 0$$

Which value is a solution to the given equation?

A) 3
B) $\sqrt{3}$
C) $-\sqrt{3}$
D) -3

21

$$T = 2P\sqrt{\frac{L}{G}}$$

The given equation relates the positive real numbers G, L, P, and T. Which equation correctly expresses G in terms of L, P, and T ?

A) $\dfrac{2PL}{T^2}$
B) $\dfrac{T^2}{4LP^2}$
C) $\dfrac{LT^2}{4P^2}$
D) $\dfrac{4LP^2}{T^2}$

22

$$x(x - 7) + 9(x - 7) = 0$$

What is the negative solution to the given equation?

23

$$x(y-2)=-y$$

The given equation relates the positive numbers x and y. Which equation correctly expresses y in terms of x ?

A) $y=\dfrac{x}{x-2}$

B) $y=\dfrac{x}{x+2}$

C) $y=\dfrac{2x}{x-1}$

D) $y=\dfrac{2x}{x+1}$

24

$$V=P(1-r)^t$$

The value V of a car depreciates over t years according to the given equation, where P is the original price and r is the annual rate of depreciation. Which of the following correctly expresses r in terms of V, P, and t ?

A) $r=1-\sqrt[t]{\dfrac{V}{P}}$

B) $r=1+\sqrt[t]{\dfrac{V}{P}}$

C) $r=\sqrt[t]{\dfrac{V}{P}}-1$

D) $r=1-\dfrac{\sqrt[t]{V}}{P}$

25

$$\frac{4}{x^2-6x+9}=9$$

If x is a solution to the given equation, which of the following could be the value of $x-3$?

A) $\dfrac{2}{3}$

B) $\dfrac{3}{2}$

C) $\dfrac{7}{3}$

D) $\dfrac{9}{2}$

26

$$\sqrt{2x+8}-3\sqrt{x-3}=0$$

What value of x is the solution to the given equation?

27

$$x-c=x^2-c^2$$

Which of the following are solutions to the given equation, where c is a positive constant?

I. c
II. $-c$
III. $1-c$

A) I only

B) II only

C) I and III only

D) I, II, and III

28

$$20-\sqrt{x}=\frac{2}{3}\sqrt{x}+10$$

If $x>0$, what value of x satisfies the given equation?

EXERCISE 2: Answers for this exercise start on page 321.

1

$$\sqrt{x-1}-2=11$$

What value of x is the solution to the given equation?

2

If $\frac{5x}{6}-4=12-\frac{7x}{6}$, what is the value of $4x$?

3

$$B+8A=5(4C+B)$$

The given equation can be rewritten as $B=mA-nC$, where m and n are constants. What is the value of $m+n$?

4

$$(2x+7)^2=25$$

What is the sum of the solutions to the given equation?

5

$$\frac{2}{3}t+m=40$$

A tennis player uses the given equation to estimate the tension t, in pounds, in his racquet strings after m tennis matches. Which of the following equations correctly expresses t in terms of m ?

A) $t=-\frac{3}{2}(40-m)$

B) $t=-\frac{3}{2}(40)-m$

C) $t=\frac{3}{2}(40-m)$

D) $t=\frac{3}{2}(40)-m$

6

$$\frac{3}{5}(x+11)-\frac{2}{3}(x+11)=-8$$

What value of x is the solution to the given equation?

7

If $\frac{4}{k+2}=\frac{x}{3}$, where $k\neq-2$, which equation correctly expresses k in terms of x ?

A) $k=\frac{12-2x}{x}$

B) $k=\frac{12+2x}{x}$

C) $k=\frac{x}{12+2x}$

D) $k=12x-2$

8

$$\sqrt{x^2 + p^2} = q$$

In the given equation, p and q are positive constants. Which of the following is one of the solutions to the given equation?

A) $-\sqrt{q^2 - p^2}$

B) $\sqrt{p^2 + q^2}$

C) $\sqrt{q - p^2}$

D) $-p^2 - q^2$

9

$$\sqrt{(x+6)(x-6)} = 8$$

What is the positive solution to the given equation?

10

$$2u = \frac{18}{v} - 6$$

The given equation relates the numbers u and v, where $u > 0$ and $v > 0$. Which equation correctly expresses v in terms of u ?

A) $v = \dfrac{9}{u+3}$

B) $v = \dfrac{u}{9} - 3$

C) $v = \dfrac{u}{9} + 3$

D) $v = \dfrac{u+3}{9}$

11

$$\frac{k}{3x + 3y} = k + 1$$

The given equation relates the distinct positive numbers x, y, and k. Which equation correctly expresses $x + y$ in terms of k ?

A) $x + y = \dfrac{k}{3k+1}$

B) $x + y = \dfrac{k}{3k+3}$

C) $x + y = \dfrac{3k}{k+1}$

D) $x + y = \dfrac{3k+3}{k}$

12

$$\left(\frac{1}{8} + \frac{1}{x}\right) t = 1$$

The given equation represents the number of days t it will take Jim and Andy to paint a 300-foot fence working together given that it would take Jim x days to paint the fence working alone. If it takes 6 days for Jim and Andy to paint the fence working together, how many days would it have taken Jim to paint the fence working alone?

13

$$\frac{a}{b}=\frac{a+1}{2c}$$

The given equation relates the distinct positive numbers a, b, and c. Which equation correctly expresses a in terms of b and c ?

A) $a=\dfrac{b}{b-2c}$

B) $a=\dfrac{b}{2c-b}$

C) $a=\dfrac{b+c}{2c}$

D) $a=\dfrac{c-b}{2c}$

14

$$\frac{3r}{t}=6\sqrt{2s}$$

The given equation relates the positive numbers r, s, and t. Which equation correctly expresses s in terms of r and t ?

A) $s=\dfrac{r^2}{2t^2}$

B) $s=\dfrac{r^2}{4t^2}$

C) $s=\dfrac{r^2}{8t^2}$

D) $s=\dfrac{3r^2}{4t^2}$

15

$$ax+7a+x+7=b$$

In the given equation, a and b are positive constants. Which equation correctly expresses $x+7$ in terms of a and b ?

A) $x+7=\dfrac{b}{a}-7$

B) $x+7=\dfrac{b}{2a}$

C) $x+7=\dfrac{b}{a+1}$

D) $x+7=\dfrac{b-1}{a}$

16

If $(x+1)(x-2)=7x-18$, what is the value of $7x-18$?

17

$$xy-1=z-xz$$

The given equation relates the positive numbers x, y, and z. Which equation correctly expresses x in terms of y and z ?

A) $\dfrac{z}{y}+1$

B) $\dfrac{z}{y+z}+1$

C) $\dfrac{z+1}{yz}$

D) $\dfrac{z+1}{y+z}$

18

If $\dfrac{2\sqrt{x+4}}{3}=6$ and $x>0$, what is the value of x ?

19

If $\dfrac{2x+y}{x-2y}=3$, what is the value of $\dfrac{x+y}{x-y}$?

20

$$\frac{x-n}{3} = x(x-n)$$

Which of the following are solutions to the given equation, where n is a positive constant greater than 1 ?

I. $\frac{1}{3}$

II. $\frac{n}{3}$

III. n

A) I and II only

B) I and III only

C) II and III only

D) I, II, and III

21

If $x + y = \sqrt{x^2 + y^2 + 16}$, what is the value of xy ?

22

$$\sqrt{x+9} = \frac{x}{2} - 3$$

What is the solution to the given equation?

23

$$\sqrt{2(x-k)} = x - k$$

In the given equation, k is a constant. Which of the following is the sum of the solutions to the given equation?

A) $2k$

B) $k+1$

C) $2k-2$

D) $2k+2$

24

$$\frac{x}{\sqrt{x+b}} = 7 - \frac{b}{\sqrt{x+b}}$$

In the given equation, b is a positive constant. Which of the following is the solution to the given equation?

A) $7-b$

B) $49-b$

C) $b-49$

D) b^2+49

25

If $\sqrt{x+2} = 2 - \sqrt{x}$, what is the value of $2 - \sqrt{x}$?

A) $\frac{1}{2}$

B) $\frac{3}{2}$

C) $\sqrt{\frac{5}{2}}$

D) 2