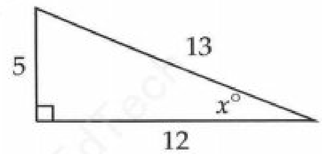
21 Trigonometry

Let's use a 5–12–13 right triangle to illustrate the functions sine, cosine, and tangent—the three trigonometric functions you'll need to know:

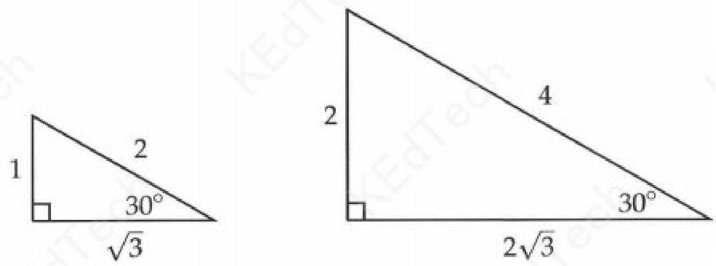


For the x° angle above,

$$\sin x^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} \qquad \cos x^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} \qquad \tan x^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$$

where "opposite" refers to the length of the side opposite the x° angle and "adjacent" refers to the length of the side next to the x° angle (the one that's not the hypotenuse).

The outputs of trigonometric functions are just ordinary numbers, so don't let the strange names confuse you. They're just ratios. For instance, the sine of 30° ($\sin 30^\circ$) is always equal to $\frac{1}{2}$. Why? Let's take a look at the 30–60–90 triangles below.



As you can see, in the triangle on the left, $\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$, and in the triangle on the right, $\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{2}{4} = \frac{1}{2}$. Because all right triangles with a 30° angle are similar, the ratios between side lengths are always the same. Therefore, $\sin 30^\circ$ is always equal to $\frac{1}{2}$.

In trigonometry, angles are often expressed in radians. For example, $\sin\frac{\pi}{6} = \frac{1}{2}$, where $\frac{\pi}{6}$ radians is 30°. On the SAT, you should assume that any trigonometric expression involving π is in radians, not degrees.

Many students over-complicate trigonometry because they view $\sin x, \cos x$, and $\tan x$ as different from regular numbers. Perhaps because of the notation, students sometimes make mistakes like the following:

$$\frac{\sin 2x}{x} = \sin 2$$

The operation above is not possible because $\sin 2x$ is one "entity." You cannot separate sin and $2x$ and treat them independently just like you can't mathematically separate $f(x)$ into f and x.

The definitions of sine, cosine, and tangent are best memorized through the acronym **SOH-CAH-TOA**, *S* for sine (opposite over hypotenuse), *C* for cosine (adjacent over hypotenuse), and *T* for tangent (opposite over adjacent).

Aside from the definitions, you should also memorize the following relationship between the sine and cosine of complementary angles (angles whose measures sum to 90°):

The sine of an angle is equal to the cosine of the complementary angle.

$$\sin x^\circ = \cos(90^\circ - x^\circ)$$

The reverse is true. The cosine of an angle is equal to the sine of the complementary angle.

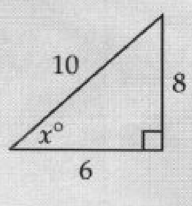
$$\cos x^\circ = \sin(90^\circ - x^\circ)$$

Expressed in radians,

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \text{and} \qquad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

where θ is the measure of the angle in radians.

EXAMPLE 1:



In the triangle shown, what is the value of $\cos x^\circ$?

A) $\frac{3}{5}$ B) $\frac{3}{4}$ C) $\frac{4}{5}$ D) $\frac{4}{3}$

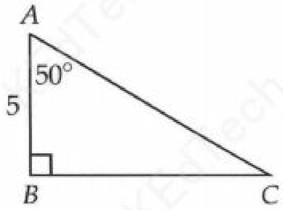
$$\cos x^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$

Answer $\boxed{(A)}$.

EXAMPLE 2: In triangle ABC, angle B is a right angle, angle A has a measure of $50°$, and $AB = 5$. Which of the following represents the length of $\overline{BC}$?

A) $5\cos 50°$ B) $\dfrac{5}{\cos 50°}$ C) $5\tan 50°$ D) $\dfrac{5}{\tan 50°}$

This is the classic trigonometry problem that illustrates the basic purpose of sine, cosine, and tangent: to help us find certain side lengths of a right triangle.



We're given the measure of angle A. The side we're given, $\overline{AB}$, is adjacent to angle A, and the side we're looking for, $\overline{BC}$, is opposite angle A. Now which of the three trig functions relates the opposite and adjacent sides? Tangent. So let's construct an equation using $\tan(A)$:

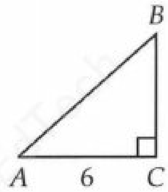
$$\tan(A) = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AB}$$

$$\tan 50° = \frac{BC}{5}$$

$$5\tan 50° = BC$$

Answer $\boxed{(C)}$. Using a calculator to get a better sense of the actual length, we get $BC = 5\tan 50° = 5.96$.

EXAMPLE 3: In triangle ABC, angle C is a right angle, $AC = 6$, and $\cos(A) = \frac{3}{4}$. The length of $\overline{BC}$ can be written as $2\sqrt{k}$, where k is a constant. What is the value of k ?



Since $\cos(A) = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{AC}{AB}$,

$$\frac{AC}{AB} = \frac{3}{4}$$

Cross-multiplying yields

$$3AB = 4AC$$

$$AB = \frac{4}{3}AC = \frac{4}{3}(6) = 8$$

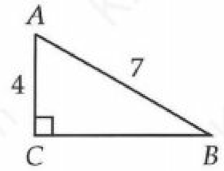
By the pythagorean theorem, $BC = \sqrt{AB^2 - AC^2} = \sqrt{8^2 - 6^2} = \sqrt{28} = 2\sqrt{7}$. Therefore, $k = \boxed{7}$.

EXAMPLE 4: In triangle ABC, $\sin(B) = \dfrac{4}{7}$ and angle C is a right angle. What is the value of $\cos(A)$?

A) $\dfrac{\sqrt{33}}{7}$ B) $\dfrac{4}{7}$ C) $\dfrac{33}{49}$ D) $\dfrac{16}{49}$

Solution 1: Since C is the right angle in the triangle, the measures of angles A and B must sum to $90°$. Since the cosine of an angle is equal to the sine of the complementary angle, $\cos(A) = \sin(90 - A) = \sin(B) = \dfrac{4}{7}$. Essentially, $\cos(A)$ is equal to $\sin(B)$, so the value of $\cos(A)$ is also $\dfrac{4}{7}$. Answer $\boxed{(B)}$.

Solution 2: Since $\sin(B) = \dfrac{4}{7}$, we can let AC, the side opposite angle B, have a length of 4 and AB, the hypotenuse, have a length of 7.



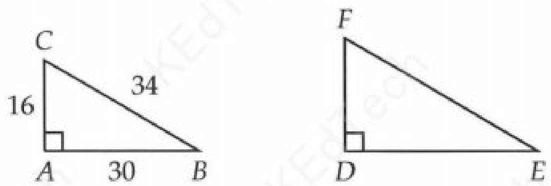
Note that these may not be the actual side lengths of triangle ABC—we're just using convenient numbers to help us derive the value of $\cos(A)$, and we're able to do so because the actual side lengths, whatever they are, must be in the same ratio. With these lengths in place, we get $\cos(A) = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{AC}{AB} = \dfrac{4}{7}$. Answer $\boxed{(B)}$. Note that $\tan(A)$ can also be found by first using the pythagorean theorem to find BC.

EXAMPLE 5:

$$AB = 30$$
$$BC = 34$$
$$AC = 16$$

The side lengths of right triangle ABC are given. Triangle ABC is similar to triangle DEF, where A corresponds to D and B corresponds to E. What is the value of $\sin(E)$?

Since BC is the longest side, it must be the hypotenuse. Angle A is then the right angle.



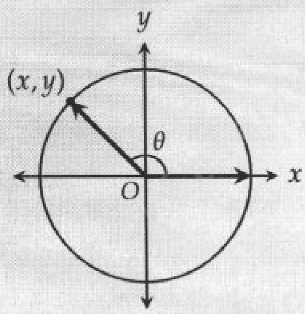
Since angle E corresponds to angle B, they must have equal measures. Thus,

$$\sin(E) = \sin(B) = \frac{AC}{BC} = \frac{16}{34} = \boxed{\frac{8}{17}}$$

As we learned at the beginning of this chapter, the sine of an angle in one right triangle is equal to the sine of the corresponding angle in a similar triangle because the ratios between side lengths are the same across similar triangles. This, of course, is true not just for sine but cosine and tangent as well.

So far, we've only looked at trig functions in the context of right triangles, which don't have any angles greater than 90 degrees. But what about angles that are greater than or equal to 90°? If sine is opposite over hypotenuse, is $\sin 135^\circ$ even possible? Yes, but only because the definitions of sine, cosine, and tangent we've been using so far are too narrow. These definitions help introduce students to trigonometry, but they are incomplete. To find the value of a trigonometric function for an obtuse angle, we have to use the following definitions:

Given an angle in standard position (its vertex is at $(0,0)$ and its initial side is along the positive x-axis) drawn on the unit circle (center at $(0,0)$ and a radius of 1),

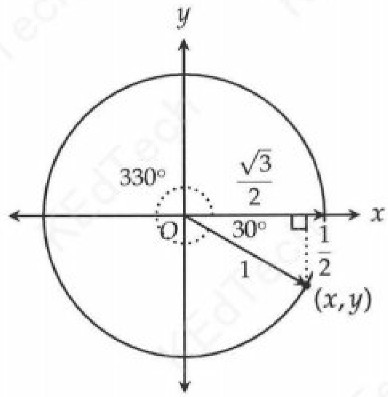


- The sine of the angle is equal to the y-coordinate of the point where the angle terminates on the unit circle.
- The cosine of the angle is equal to the x-coordinate of the point where the angle terminates on the unit circle.
- The tangent of the angle is equal to the sine of the angle divided by the cosine of the angle (the y-coordinate divided by the x-coordinate).

Let's do a couple of examples.

1. What are the values of $\sin 330^\circ$, $\cos 330^\circ$, and $\tan 330^\circ$?

   First, draw the unit circle and figure out which quadrant the angle terminates in. In this case, 330° is almost a full circle and terminates in the fourth quadrant.



   The terminal side of the angle forms a $360^\circ - 330^\circ = 30^\circ$ angle with the x-axis. This acute angle is sometimes called the **reference angle**. To find the coordinates of the desired point on the unit circle, draw a line from the point to the x-axis to form a right triangle. Since the hypotenuse of this right triangle is a radius of the unit circle, its length is 1. We therefore have a 30–60–90 triangle with a hypotenuse of 1. As we've already learned, the side opposite 30° (the shorter leg) is half as long as the hypotenuse, and the side opposite 60° (the longer leg) is $\sqrt{3}$ times longer than the shorter leg. Thus, the

y-coordinate is $-\frac{1}{2}$ (negative because the point is in the fourth quadrant) and the x-coordinate is $\frac{\sqrt{3}}{2}$.

The values of the three trig functions are then $\sin 330^\circ = -\frac{1}{2}$, $\cos 330^\circ = \frac{\sqrt{3}}{2}$, and

$\tan 330^\circ = \frac{\sin 330^\circ}{\cos 330^\circ} = -\frac{1}{2} \div \frac{\sqrt{3}}{2} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$. Rationalizing the denominator, we get

$\tan 330^\circ = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

2. What are the values of $\sin \frac{35\pi}{4}$, $\cos \frac{35\pi}{4}$, and $\tan \frac{35\pi}{4}$?

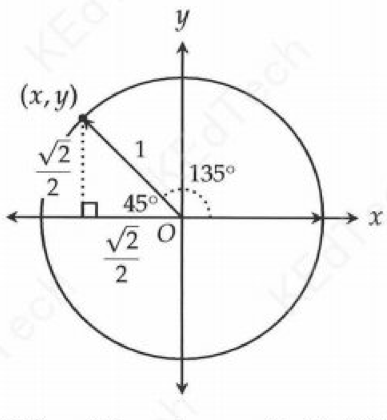
Since the angle is in terms of π, we should assume radian measure. To make this example easier to follow, we'll convert the angle to degrees:

$$\frac{35\pi}{4} \text{ radians} \times \frac{180^\circ}{\pi \text{ radians}} = 1{,}575^\circ$$

Since angles that are 360° (2π radians) apart terminate at the same point, we can simplify this problem by continually subtracting 360° (2π radians) from the angle until we get a resulting angle less than 360° (2π radians). In this case, $1{,}575 \div 360 \approx 4$ multiples of 360° go into $1{,}575^\circ$, so for the purposes of this question,

$$1{,}575^\circ = 1{,}575^\circ - 4 \times 360^\circ = \underline{135^\circ}$$

which terminates in the 2nd quadrant.



Since the reference angle is $180^\circ - 135^\circ = 45^\circ$, we have a 45–45–90 triangle with a hypotenuse of 1. In any 45–45–90 triangle, the hypotenuse is $\sqrt{2}$ times longer than the legs. Therefore, the length of each leg in the triangle above is $\frac{1}{\sqrt{2}}$, or $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, when the denominator is rationalized.

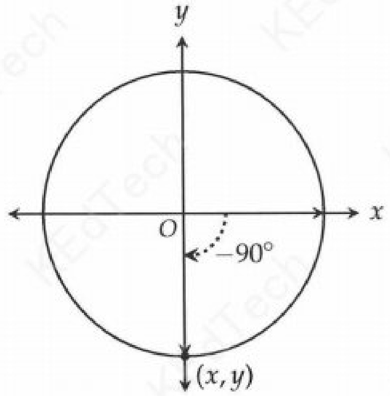
The y-coordinate of the desired point is then $\frac{\sqrt{2}}{2}$ and the x-coordinate is $-\frac{\sqrt{2}}{2}$ (negative because the point is in the second quadrant). Given these coordinates, the values of the three trig functions are

$\sin \frac{35\pi}{4} = \sin 135^\circ = \frac{\sqrt{2}}{2}$, $\cos \frac{35\pi}{4} = \cos 135^\circ = -\frac{\sqrt{2}}{2}$, and

$\tan \frac{35\pi}{4} = \tan 135^\circ = \frac{\sin 135^\circ}{\cos 135^\circ} = \frac{\sqrt{2}}{2} \div -\frac{\sqrt{2}}{2} = -1$.

3. What are the values of $\sin\left(-\frac{\pi}{2}\right)$, $\cos\left(-\frac{\pi}{2}\right)$, and $\tan\left(-\frac{\pi}{2}\right)$?

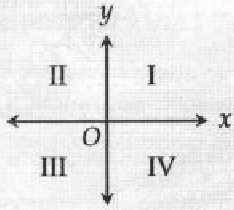
$-\frac{\pi}{2}$ radians is equivalent to $-90°$, or $-90° + 360° = 270°$. Let's look at the unit circle.



In this case, no special right triangle is needed. Since the unit circle has a radius of 1, it's obvious that the coordinates of the intercepted point are $(0, -1)$. Therefore, $\sin\left(-\frac{\pi}{2}\right) = -1$, $\cos\left(-\frac{\pi}{2}\right) = 0$, and $\tan\left(-\frac{\pi}{2}\right) = \frac{-1}{0} = \text{undefined}$.

Desmos Note: The process we've outlined for finding trig values by hand only works when the angle terminates at a point whose coordinates are obvious or can be found using special right triangle relationships. For all other angles, such as 88°, you would have to use a calculator. And practically speaking, you should always use Desmos to find trig values on the SAT, not the unit circle. Even when the answer choices are exact values such as $\frac{\sqrt{2}}{2}$, it's still easier to compare your decimal answer from Desmos with the decimal value of each answer choice. Why did we cover the unit circle in this chapter then? Because it improves your understanding of trigonometry and illustrates concepts that could be helpful for certain SAT questions.

Lastly, when determining the value of a trig function, it's good to know beforehand what the sign of the result will be (positive or negative):



- Sine, cosine, and tangent are all positive in the first quadrant ($0° < \theta < 90°$).
- Only sine is positive in the second quadrant ($90° < \theta < 180°$).
- Only tangent is positive in the third quadrant ($180° < \theta < 270°$).
- Only cosine is positive in the fourth quadrant ($270° < \theta < 360°$).

This information is best memorized using the acronym **ASTC** (All Students Take Calculus). All the functions are positive in the first quadrant, only sine is positive in the second, and so on.

EXERCISE 1: Answers for this exercise start on page 394.

1

In triangle ABC, angle C is a right angle, $AC = 3$, and $BC = 4$. What is the value of $\tan(A)$?

A) $\frac{3}{4}$

B) $\frac{4}{5}$

C) $\frac{4}{3}$

D) $\frac{5}{3}$

2

In triangle ABC, the measure of angle C is 90° and the length of $\overline{AB}$ is 30. If $\cos(A) = \frac{5}{6}$, what is the length of $\overline{AC}$?

3

In triangle ABC, the measure of angle C is 90° and $\sin(A) = \frac{12}{13}$. What is the value of $\cos(B)$?

A) $\frac{5}{13}$

B) $\frac{12}{13}$

C) $\frac{13}{12}$

D) $\frac{13}{5}$

4

In triangle DEF, the measure of angle F is 64°, angle E is a right angle, and the length of $\overline{DF}$ is 15. Which of the following represents the length of $\overline{EF}$?

A) $15\cos 26^\circ$

B) $15\cos 64^\circ$

C) $\frac{15}{\cos 26^\circ}$

D) $\frac{15}{\cos 64^\circ}$

5

Triangle ABC is similar to triangle DEF, where angle A corresponds to angle D and angles C and F are right angles. If $\sin(B) = 0.352$, what is the value of $\sin(E)$?

6

$$AB = 198$$
$$BC = 390$$
$$AC = 336$$

The side lengths of right triangle ABC are given. Triangle ABC is similar to triangle PQR, where A corresponds to P and B corresponds to Q. What is the value of $\cos(Q)$?

A) $\frac{33}{65}$

B) $\frac{33}{56}$

C) $\frac{56}{65}$

D) $\frac{56}{33}$

7

In right triangle ABC, the measures of angle A and angle C sum to 90 degrees. The value of $\cos(A) = \frac{24}{25}$. What is the value of $\sin(C)$?

8

In triangle DEF, $\sin(E) = \frac{24}{74}$ and angle D is a right angle. What is the value of $\sin(F)$?

9

Triangle ABC is similar to triangle DEF, where angle A corresponds to angle D and angles C and F are right angles. If $\tan(B) = 4$, what is the value of $\tan(D)$?

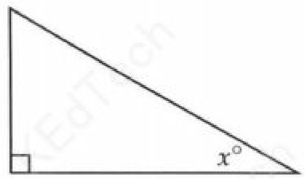
10

Triangle JKL is similar to triangle UVW, where angle J corresponds to angle U and angles K and V are right angles. If $\cos J = \frac{65}{97}$ and $\cos L = \frac{72}{97}$, what is the value of $\tan W$?

11

Triangle ABC is similar to triangle RST, where angle A corresponds to angle R and angles B and S are right angles. The length of each side of triangle ABC is 2 times the length of the corresponding side of triangle RST. If $\tan(C) = \frac{28}{45}$, what is the value of $\sin(T)$?

12



Note: Figure not drawn to scale.

In the triangle shown, $\cos(90° - x°) = \frac{8}{17}$. What is the value of $\cos x°$?

A) $\frac{8}{15}$

B) $\frac{17}{15}$

C) $\frac{8}{17}$

D) $\frac{15}{17}$

13

Triangle FGH is similar to triangle LMN, where F corresponds to L and H corresponds to N. Angles H and N are right angles. If $\cos(F) = \frac{2}{3}$ and $LM = 18$, what is the length of $\overline{MN}$?

A) 12

B) $6\sqrt{5}$

C) 15

D) $9\sqrt{5}$

EXERCISE 2: Answers for this exercise start on page 396.

1

In triangle ABC, the measure of angle A is $25°$, angle C is a right angle, and the length of $\overline{BC}$ is 16. Which of the following represents the length of $\overline{AC}$?

A) $16\sin 25°$

B) $16\tan 25°$

C) $\dfrac{16}{\sin 25°}$

D) $\dfrac{16}{\tan 25°}$

2

In triangle ABC, the measure of angle A is $40°$ and B is a right angle. Which of the following additional pieces of information is sufficient to determine the length of $\overline{AB}$?

I. The length of $\overline{AC}$
II. The length of $\overline{BC}$

A) I is sufficient, but II is not.

B) II is sufficient, but I is not.

C) I is sufficient and II is sufficient.

D) Neither I nor II is sufficient.

3

In a right triangle, the sine of one of the two acute angles is $\dfrac{\sqrt{3}}{2}$. What is the sine of the other acute angle?

A) $\dfrac{1}{2}$

B) $\dfrac{\sqrt{3}}{2}$

C) $\dfrac{1}{\sqrt{3}}$

D) $\dfrac{\sqrt{2}}{2}$

4

What is the value of $\sin\dfrac{76\pi}{3}$?

A) $\dfrac{\sqrt{3}}{2}$

B) $\dfrac{1}{2}$

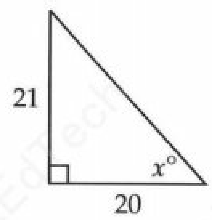
C) $-\dfrac{1}{2}$

D) $-\dfrac{\sqrt{3}}{2}$

5

In right triangle RST, $\cos T = \sin 52°$. What is the measure, in degrees, of angle T ?

6



Note: Figure not drawn to scale.

In the triangle shown, what is the value of $\tan(90° - x°)$?

A) $\dfrac{20}{29}$

B) $\dfrac{20}{21}$

C) $\dfrac{21}{29}$

D) $\dfrac{21}{20}$

7

The value of $\tan \frac{49\pi}{6}$ can be written as $k\sqrt{3}$, where k is a constant. What is the value of k ?

A) $\frac{1}{3}$

B) $\frac{1}{\sqrt{3}}$

C) 1

D) 3

8

Triangle LMN is similar to triangle QRS, where L corresponds to Q and angles M and R are right angles. If $\tan(N) = \frac{5}{9}$ and $QR = 4$, what is the length of $\overline{RS}$?

9

Triangle JKL is similar to triangle RST, where angle J corresponds to angle R and angles K and S are right angles. If $\cos(L) = \frac{119}{169}$, what is the value of $\tan(T)$?

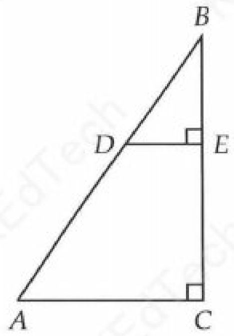
A) $\frac{120}{169}$

B) $\frac{119}{120}$

C) $\frac{120}{119}$

D) $\frac{169}{119}$

10



Note: Figure not drawn to scale.

In right triangle ABC shown, $DE = 10$ and $\tan(A) = \frac{5}{4}$. What is the length of $\overline{BE}$?

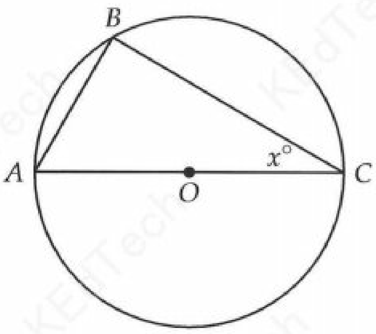
11

Triangle ABC is similar to triangle DEF, where angle A corresponds to angle D and angles B and E are right angles. If $AB = 2$, $DF = 12\sqrt{2}$, and $\tan(A) = \sqrt{7}$, what is the length of $\overline{DE}$?

12

In triangle ABC, the measure of angle A is equal to the measure of angle C. If $\cos A = \frac{3}{8}$ and $AB = 20$, what is the length of $\overline{AC}$?

13



The circle shown has center O and diameter $\overline{AC}$. Point B lies on the circle. If $AC = 1$, which of the following represents the area of triangle ABC ?

A) $\dfrac{\tan x}{2}$

B) $2\sin x$

C) $\dfrac{\sin x \cos x}{2}$

D) $\dfrac{\sin x \tan x}{2}$