2
Expressions

Algebraic expressions are just combinations of numbers and variables. Both $x^2 + y$ and $\frac{3m-k}{2}$ are examples of expressions. In this chapter, we'll cover some fundamental techniques that will allow you to deal with questions involving expressions quickly and effectively.

1. Combining Like Terms

When adding or subtracting, be careful of terms that look like they can go together but can't. For example, you cannot combine $b^2 + b$ to make b^3, nor can you combine $a + ab$ to make $2ab$. For an addition or subtraction to be valid, the variables in each term have to exactly match.

EXAMPLE 1: Which expression is equivalent to $2(2a^2 - 3a^2b^2 - 4b^2) - (a^2 + 5a^2b^2 - 10b^2)$?

A) $-6a^2b^2$ B) $3a^2 - 11a^2b^2 - 18b^2$ C) $3a^2 - 11a^2b^2 + 2b^2$ D) $5a^2 + 2a^2b^2 + 2b^2$

$$\begin{aligned} 2(2a^2 - 3a^2b^2 - 4b^2) - (a^2 + 5a^2b^2 - 10b^2) &= 4a^2 - 6a^2b^2 - 8b^2 - a^2 - 5a^2b^2 + 10b^2 \\ &= 3a^2 - 11a^2b^2 + 2b^2 \end{aligned}$$

Answer $\boxed{\text{(C)}}$.

2. Simplifying Fractional Expressions

Fractional expressions can be simplified when 1) the numerator is either a product of expressions or a common factor itself, 2) the denominator is either a product of expressions or a common factor itself, and 3) there is a factor common to both the numerator and denominator.

Some examples:

- $\frac{4xy(x+y)(x-y)}{6x(x-y)}$ can be simplified because both the numerator and denominator are products that share common factors of 2, x, and $x - y$: $\frac{{}^{2}\cancel{4}\cancel{x}y(x+y)\cancel{(x-y)}}{{}^{3}\cancel{6}\cancel{x}\cancel{(x-y)}} = \frac{2y(x+y)}{3}$.

- $\dfrac{4x}{4y+1}$ cannot be simplified because the denominator is neither a product of expressions nor a common factor itself. Doing the following would be incorrect: $\dfrac{\cancel{4}x}{\cancel{4}y+1}$. Similarly, $\dfrac{4y+1}{4x}$ cannot be simplified either.
- $\dfrac{4(x+1)}{x+1}$ can be simplified because the denominator itself is a common factor: $\dfrac{4\cancel{(x+1)}}{\cancel{x+1}} = 4$.
- $\dfrac{x+8}{x+4}$ cannot be simplified because neither the numerator nor the denominator is a product of expressions or a common factor. Doing the following would be incorrect: $\dfrac{x+\cancel{8}^{2}}{x+\cancel{4}^{1}} = \dfrac{x+2}{x+1}$.
- $\dfrac{8x}{4xy+4x}$ seems like it cannot be simplified because the denominator is neither a product nor a common factor itself. However, the denominator can be factored into a product: $\dfrac{8x}{4x(y+1)}$. And now we can simplify: $\dfrac{{}^{2}\cancel{8x}}{\cancel{4x}(y+1)} = \dfrac{2}{y+1}$.

3. Expansion and Factoring

EXAMPLE 2: Which expression is equivalent to $2(x-4)(2x+3)$?

A) $4x^2 - 10x - 24$ B) $4x^2 + 10x - 24$ C) $4x^2 + 10x + 24$ D) $8x^2 - 20x - 24$

Binomials are expressions with only two terms, such as the $x-4$ and $2x+3$ in this example. To get started, we can distribute the "2" to either the first binomial or the second, but not both. Let's distribute it to the first binomial:

$$2(x-4)(2x+3) = (2x-8)(2x+3)$$

Now we can use a method called FOIL (first, outer, inner, last) to expand this product of two binomials. If you haven't heard of FOIL, it's the same as distributing each term.

$$(2x-8)(2x+3)$$

First: $2x \cdot 2x = \boxed{4x^2}$, Outer: $2x \cdot 3 = \boxed{6x}$, Inner: $-8 \cdot 2x = \boxed{-16x}$, Last: $-8 \cdot 3 = \boxed{-24}$

Putting these results together, we get

$$(2x-8)(2x+3) = 4x^2 + 6x - 16x - 24 = 4x^2 - 10x - 24$$

Answer $\boxed{(A)}$.

EXAMPLE 3: Which expression is equivalent to $9x^3y + 6x^2y^2$?

A) $3x^2y(3x + 2y)$ B) $3x^2y(3x + 2y^2)$ C) $3x^2y^2(3x + 2y)$ D) $3x^2y^2(3x + 2y^2)$

We can factor this expression using the greatest common factor of both terms, $3x^2y$:

$$9x^3y + 6x^2y^2 = 3x^2y(3x + 2y)$$

Answer $\boxed{(A)}$.

EXAMPLE 4: Which expression is equivalent to $\dfrac{2x(x+4) - 3(x+4)}{4x-6}$, where $x > \dfrac{3}{2}$?

A) $\dfrac{x+4}{2}$ B) $\dfrac{2x-3}{4}$ C) $\dfrac{2x^2 - 3x - 12}{4x-6}$ D) $\dfrac{2x^2 + 5x - 8}{4x-6}$

Don't be too quick to expand the numerator. Notice that both terms have a $(x + 4)$ in common. We can factor out this $(x + 4)$ from the terms in the numerator and a 2 from the terms in the denominator:

$$\frac{2x(x+4) - 3(x+4)}{4x-6} = \frac{(x+4)(2x-3)}{4x-6} = \frac{(x+4)(2x-3)}{2(2x-3)} = \frac{(x+4)\cancel{(2x-3)}}{2\cancel{(2x-3)}} = \frac{x+4}{2}$$

Answer $\boxed{(A)}$.

If you're confused at all about how we factored the numerator, another way to look at it is to let $A = (x + 4)$. Then the numerator becomes $2xA - 3A = A(2x - 3) = (x + 4)(2x - 3)$.

In addition to the techniques shown in the previous examples, there are three essential patterns you must know, summarized by the formulas below:

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $a^2 - b^2 = (a + b)(a - b)$

Memorize these forwards and backwards. They show up very often.

EXAMPLE 5: Which expression is equivalent to $16x^4 - 81$?

A) $(4x^2 + 9)^2$ B) $(2x + 3)^2(2x - 3)^2$ C) $(4x^2 + 27)(4x^2 - 3)$ D) $(4x^2 + 9)(2x + 3)(2x - 3)$

Here, we have an expression of the form $a^2 - b^2$, a difference of two squares, where $a = \sqrt{16x^4} = 4x^2$ and $b = \sqrt{81} = 9$. Since $a^2 - b^2 = (a + b)(a - b)$,

$$16x^4 - 81 = (4x^2 + 9)(4x^2 - 9)$$

But wait! We're not done. In this result is another difference of two squares, $4x^2 - 9$. We have to factor this as well.

$$(4x^2 + 9)(4x^2 - 9) = (4x^2 + 9)(2x + 3)(2x - 3)$$

Answer $\boxed{(D)}$.

As you can see, memorizing a formula is not enough. You have to know when and how to apply it. The more you do these types of problems, the better you'll get at recognizing these patterns and their variations.

EXAMPLE 6: Which expression is equivalent to $(3x-5)^{\frac{2}{3}}$?

A) $(9x^2+25)^3$ B) $\sqrt[3]{9x^2+25}$ C) $\sqrt[3]{9x^2-15x+25}$ D) $\sqrt[3]{9x^2-30x+25}$

As we learned in the chapter on exponents, the numerator of a fractional exponent is the power and the denominator is the root. Here's the step-by-step simplification:

$$(3x-5)^{\frac{2}{3}} = [(3x-5)^2]^{\frac{1}{3}} = \sqrt[3]{(3x-5)^2}$$

We now have an expression of the form $(a-b)^2$ inside the root, where $a = 3x$ and $b = 5$. Using the formula $(a-b)^2 = a^2 - 2ab + b^2$ to expand it, we get

$$\sqrt[3]{(3x-5)^2} = \sqrt[3]{(3x)^2 - 2(3x)(5) + (5)^2} = \sqrt[3]{9x^2-30x+25}$$

Answer $\boxed{(D)}$.

EXAMPLE 7: Which expression is equivalent to $16x^4 - 8x^2y^2 + y^4$?

A) $(4x^2+y^2)^2$ B) $(2x-y)^4$ C) $(2x+y)^2(2x-y)^2$ D) $(4x+y)^2(x-y)^2$

The given expression is of the form $a^2 - 2ab + b^2$, where $a = 4x^2$ and $b = y^2$. Using the formula $a^2 - 2ab + b^2 = (a-b)^2$ to factor it, we get

$$16x^4 - 8x^2y^2 + y^4 = (4x^2 - y^2)^2$$

But this result is not in the answer choices. We have to take it one step further—the expression inside the parentheses is a difference of two squares, which we know how to factor:

$$(4x^2-y^2)^2 = [(2x+y)(2x-y)]^2 = (2x+y)^2(2x-y)^2$$

Answer $\boxed{(C)}$.

4. Combining Fractions

When you're adding simple fractions,

$$\frac{1}{3} + \frac{1}{4}$$

the first step is to find the least common multiple of the denominators. We do this so that we can get a common denominator. In a lot of cases, it's just the product of the denominators, as it is here, $3 \times 4 = 12$.

$$\frac{1}{3} + \frac{1}{4} = \frac{1}{3} \cdot \frac{4}{4} + \frac{1}{4} \cdot \frac{3}{3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

Now when we're adding fractions with expressions in the denominator, the idea is the same.

EXAMPLE 8: Which expression is equivalent to $\dfrac{1}{x+2}+\dfrac{2}{x-2}$, where $x \neq -2$ and $x \neq 2$?

A) $\dfrac{3x-2}{(x+2)(x-2)}$ B) $\dfrac{3x+2}{(x+2)(x-2)}$ C) $\dfrac{3}{(x+2)(x-2)}$ D) $\dfrac{2}{(x+2)(x-2)}$

The common denominator is just the product of the two denominators: $(x+2)(x-2)$. With this in mind, we can multiply the top and bottom of each fraction by the factor they don't have:

$$\begin{aligned}\frac{1}{x+2}+\frac{2}{x-2}=\frac{1}{x+2}\cdot\frac{x-2}{x-2}+\frac{2}{x-2}\cdot\frac{x+2}{x+2}=\frac{x-2}{(x+2)(x-2)}+\frac{2(x+2)}{(x+2)(x-2)}&=\frac{(x-2)+2(x+2)}{(x+2)(x-2)}\\&=\frac{3x+2}{(x+2)(x-2)}\end{aligned}$$

Answer $\boxed{(B)}$.

EXAMPLE 9: Which expression is equivalent to $\dfrac{6}{x+3}+\dfrac{18}{x^2+3x}$, where $x \neq -3$ and $x \neq 0$?

A) $\dfrac{6}{x}$ B) $\dfrac{24}{x(x+3)}$ C) $\dfrac{30x+72}{x(x+3)}$ D) $\dfrac{24}{x^2+4x+3}$

First, factor out an x from the denominator of the second fraction to get $\dfrac{6}{x+3}+\dfrac{18}{x(x+3)}$. Now we need a common denominator to add these two fractions. Both denominators already have a factor of $(x+3)$, so multiplying them to get a common denominator would be overdoing it. All we need for the first denominator to match the second is a factor of x, so let's just multiply the top and bottom of the first fraction by x:

$$\frac{6}{x+3}+\frac{18}{x(x+3)}=\frac{6}{x+3}\cdot\frac{x}{x}+\frac{18}{x(x+3)}=\frac{6x}{x(x+3)}+\frac{18}{x(x+3)}=\frac{6x+18}{x(x+3)}=\frac{6\cancel{(x+3)}}{x\cancel{(x+3)}}=\frac{6}{x}$$

Answer $\boxed{(A)}$.

5. Flipping (Dividing) Fractions

What's the difference between $\dfrac{\frac{1}{2}}{3}$ and $\dfrac{1}{\frac{2}{3}}$?

The difference is where the longer fraction line is. The first is $\dfrac{1}{2}$ divided by 3. The second is 1 divided by $\dfrac{2}{3}$. They're not the same.

$$\frac{\frac{1}{2}}{3}=\frac{1}{2}\div 3=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$$

$$\frac{1}{\frac{2}{3}}=1\div\frac{2}{3}=1\times\frac{3}{2}=\frac{3}{2}$$

To summarize, when there is a fraction $\frac{a}{b}$ in the denominator, we flip a and b: $\frac{c}{\frac{a}{b}} = \frac{bc}{a}$.

When there is a fraction $\frac{a}{b}$ in the numerator, a and b remain where they are: $\frac{\frac{a}{b}}{c} = \frac{a}{bc}$.

EXAMPLE 10:

$$\frac{6}{\left(\frac{24}{x}\right)}$$

Which of the following is equivalent to the given expression?

A) $4x$ B) $\frac{x}{4}$ C) $\frac{1}{4x}$ D) $\frac{4}{x}$

$$\frac{6}{\left(\frac{24}{x}\right)} = \frac{6x}{24} = \frac{x}{4}$$

Answer $\boxed{(B)}$.

6. Splitting Fractions

EXAMPLE 11: Which expression is equivalent to $\frac{30+c}{6}$?

A) $\frac{5+c}{6}$ B) $\frac{10+c}{2}$ C) $5+c$ D) $5+\frac{c}{6}$

We can split the fraction into two:

$$\frac{30+c}{6} = \frac{30}{6} + \frac{c}{6} = 5 + \frac{c}{6}$$

Answer $\boxed{(D)}$. As you can see, this process reverses fraction addition.

Note that while you can split up the numerators of fractions, you cannot do so with denominators. So,

$$\frac{3}{x+y} \neq \frac{3}{x} + \frac{3}{y}$$

In fact, you cannot break up a fraction like $\frac{3}{x+y}$ any further.

EXERCISE 1: Answers for this exercise start on page 316.

1

Which expression is equivalent to $6x^2y + 6xy^2$?

A) $6xy(x + y)$
B) $12xy(x + y)$
C) $6x^2y^2(y + x)$
D) $12x^3y^3$

2

Which expression is equivalent to
$(3x^3 + 8x^2 - 4x) + (7x^2 - 11x - 7)$?

A) $3x^3 + x^2 - 15x - 7$
B) $3x^3 + 15x^2 - 15x - 7$
C) $10x^5 - 7x - 7$
D) $15x^4 + 3x^3 - 15x^2 - 7$

3

Which expression is equivalent to $\frac{1}{a} + \frac{3}{4}$, where $a \neq 0$?

A) $\frac{3 + 4a}{4a}$
B) $\frac{4 + 3a}{4a}$
C) $\frac{7}{4a}$
D) $\frac{4}{a + 4}$

4

The expression $(15x + 34)(-8x + 3)$ is equivalent to the expression $ax^2 + bx + c$, where a, b, and c are constants. What is the value of b ?

5

Which expression is equivalent to $\frac{4 + 8x}{12x}$, where $x \neq 0$?

A) $\frac{1 + 8x}{3x}$
B) $\frac{4 + 2x}{3x}$
C) $\frac{1 + 2x}{3x}$
D) 1

6

Which expression is equivalent to $484x^2 - 324$?

A) $(18x - 22)(18x + 22)$
B) $(18x - 22)(18x - 22)$
C) $(22x - 18)(22x + 18)$
D) $(22x - 18)(22x - 18)$

7

Which expression is equivalent to
$(x^2 + y)(y + z)$?

A) $x^2z + y^2 + yz$
B) $x^2y + x^2z + y^2 + yz$
C) $x^2y + y^2 + x^2z$
D) $x^2 + x^2z + y^2 + yz$

8

The expression $\frac{15x}{36x^2 + 18x}$ is equivalent to $\frac{5}{k(2x + 1)}$, where k is a constant and $x > 0$. What is the value of k ?

A) 3
B) 6
C) 12
D) 18

9

Which expression is equivalent to $3x^4 - 3$?

A) $3(x^2+1)^2$
B) $3(x^2-1)^2$
C) $3(x^3-1)(x+1)$
D) $3(x^2+1)(x+1)(x-1)$

10

Which expression is equivalent to $\dfrac{\left(\dfrac{3}{x-1}\right)}{6x-6}$, where $x \neq 1$?

A) $\dfrac{1}{2}$
B) 18
C) $\dfrac{1}{2x-2}$
D) $\dfrac{1}{2(x-1)^2}$

11

Which expression is equivalent to $2(3x^2-5)^2-(3x^2-5)^2$?

A) $9x^4-15x^2+75$
B) $9x^4-30x^2+25$
C) $9x^4-60x^2+75$
D) $18x^4-90x^2+25$

12

Which expression is equivalent to $\dfrac{xy-x^2}{xy-y^2}$, where $y \neq 0$ and $x \neq y$?

A) $-\dfrac{y}{x}$
B) $\dfrac{y}{x}$
C) $\dfrac{x}{y}$
D) $-\dfrac{x}{y}$

13

Which expression is equivalent to $\dfrac{x^2(2x-3)-(2x-3)}{x-1}$, where $x > 1$?

A) $2x^2-3x$
B) $(2x-3)(x+1)$
C) $\dfrac{2x^2-3}{x-1}$
D) $\dfrac{2x^3-5x^2+3}{x-1}$

14

Which expression is equivalent to $\dfrac{3}{3x-14}-\dfrac{1}{x-3}$?

A) $\dfrac{2}{2x-11}$
B) $-\dfrac{23}{(3x-14)(x-3)}$
C) $\dfrac{5}{(3x-14)(x-3)}$
D) $-\dfrac{3x-10}{(3x-14)(x-3)}$

15

Which expression is equivalent to $x^2(x+2)(x-2)+4$?

A) $(x^2-2)^2$
B) $(x^2+2)^2$
C) $(x-1)^2(x+2)^2$
D) $(x+1)^2(x-2)^2$

EXERCISE 2: Answers for this exercise start on page 317.

1

Which expression is equivalent to $3xy^6 + 3x^3y^3$?

A) $3xy^3(y^2 + x^2)$

B) $3xy^3(y^2 + x^3)$

C) $3xy^3(y^3 + x^2)$

D) $3xy^3(y^3 + x^3)$

2

Which expression is equivalent to $(5a + 3\sqrt{a}) - (2a + 5\sqrt{a})$?

A) $-2a\sqrt{a}$

B) $a\sqrt{a}$

C) $3a - 2\sqrt{a}$

D) $3a + 8\sqrt{a}$

3

Which expression is equivalent to $\frac{2}{x} - \frac{1}{3x}$, where $x > 0$?

A) $-\frac{1}{2x}$

B) $\frac{2}{x}$

C) $\frac{2}{3x}$

D) $\frac{5}{3x}$

4

Which expression is equivalent to $(x - 8)(5 - 3x^2)$?

A) $-3x^3 - 6x^2 + 24$

B) $-3x^3 + 11x^2 + 5x - 40$

C) $-3x^3 + 24x^2 + 5x - 40$

D) $-3x^3 + 29x^2 + 24$

5

Which expression is equivalent to $\frac{x^2 - 3}{x + \sqrt{3}}$, where $x \neq -\sqrt{3}$?

A) $\frac{x\sqrt{3}}{3}$

B) $x - 1$

C) $x - \sqrt{3}$

D) $x + \sqrt{3}$

6

$$(4x^2 + 4x - 1)(x^2 - 3x)$$

If the given expression is rewritten in the form $ax^4 + bx^3 + cx^2 + dx$, where a, b, c, and d are constants, what is the value of $b + c$?

7

Which expression is equivalent to $5b - \frac{b + k}{b}$, where $b > 0$?

A) $\frac{4b - k}{b}$

B) $\frac{4b + k}{b}$

C) $\frac{5b^2 - b - k}{b}$

D) $\frac{5b^2 - b + k}{b}$

8

The expression $\dfrac{(6x+8)(3x+6)}{12x+24}$ is equivalent to $\dfrac{ax+b}{2}$, where a and b are constants and $x>0$. What is the value of $a+b$?

9

Which expression is equivalent to $(3\sqrt{a}-\sqrt{b})^{\frac{2}{7}}$, where $a>b$ and $b>0$?

A) $\sqrt[7]{9a-b}$
B) $\sqrt[7]{9a-6ab+b}$
C) $\sqrt[7]{9a-3\sqrt{ab}+b}$
D) $\sqrt[7]{9a-6\sqrt{ab}+b}$

10

Which expression is equivalent to x^4-8x^2+16 ?

A) $(x-2)^4$
B) $(x-2)^3(x+2)$
C) $(x-2)^2(x+2)^2$
D) $(x^2+4)(x+2)(x-2)$

11

Which expression is equivalent to $\dfrac{y-3xy}{2x^2-2xy}+\dfrac{1+3y}{x-y}$?

A) $\dfrac{2x-3xy+4y}{2x^2-2xy}$

B) $\dfrac{2x+3xy+y}{2x^2-2xy}$

C) $\dfrac{2x+3xy+y}{2x^3-4x^2y+2xy^2}$

D) $\dfrac{4y-3xy+1}{2x^3-4x^2y+2xy^2}$

12

The expression $8x^2-\frac{1}{2}y^2$ is equivalent to $8(x-cy)(x+cy)$, where c is a positive constant. What is the value of c ?

A) $\dfrac{1}{16}$

B) $\dfrac{1}{8}$

C) $\dfrac{1}{4}$

D) $\dfrac{\sqrt{2}}{4}$

13

Which expression is equivalent to $(x+1)^2+2(x+1)(y+1)+(y+1)^2$?

A) $(x+y+1)^2$
B) $(x+y+2)^2$
C) $(x+y)^2+2$
D) $(x+y)^2-x-y$

14

Which expression is equivalent to $\dfrac{x}{x-2}+\dfrac{x}{2(2-x)}$, where $x\neq 2$?

A) $-\dfrac{x}{x-2}$

B) $-\dfrac{x}{2(x-2)}$

C) $\dfrac{x}{2(x-2)}$

D) $\dfrac{3x}{2(x-2)}$