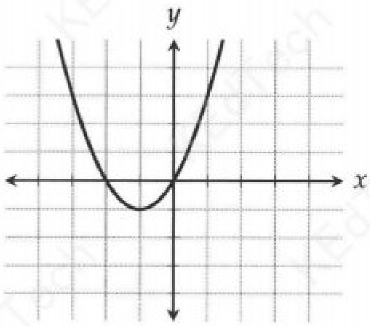
15 Function Transformations

Function transformations are changes we can make to the equation of a function to "transform" its graph in specific ways. The transformations you might encounter on the SAT are vertical shift (a.k.a. vertical translation) and horizontal shift (a.k.a. horizontal translation).

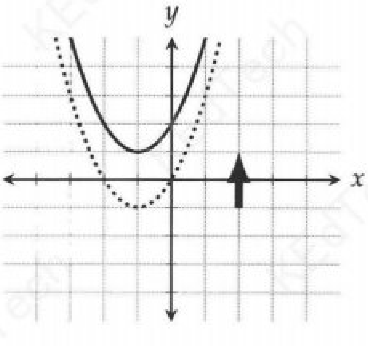
Let's start with the example function $f(x) = x^2 + 2x$, whose graph looks like



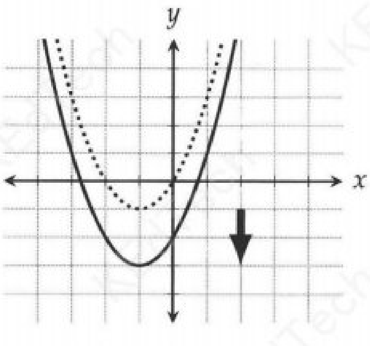
To shift the graph of $f(x)$ up, add a constant to $f(x)$. For example, $y = f(x) + 2 = x^2 + 2x + 2$ produces a graph that is 2 units above the graph of $y = f(x)$.

To shift the graph of $f(x)$ down, subtract a constant from $f(x)$. For example, $y = f(x) - 2 = x^2 + 2x - 2$ produces a graph that is 2 units below the graph of $y = f(x)$.

$y = f(x) + 2$

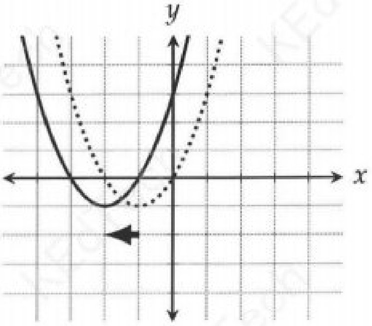


$y = f(x) - 2$

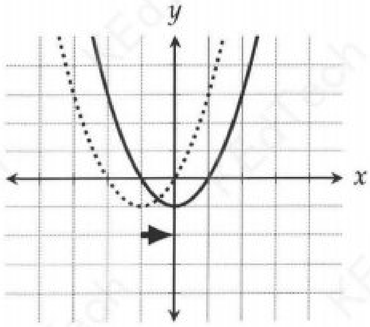


To shift the graph of $f(x)$ to the left by a units, substitute $x+a$ in for x. For example, $y = f(x+1) = (x+1)^2 + 2(x+1)$ produces a graph that is 1 unit to the left of the graph of $y = f(x)$.
To shift the graph of $f(x)$ to the right by a units, substitute $x-a$ in for x. For example, $y = f(x-1) = (x-1)^2 + 2(x-1)$ produces a graph that is 1 unit to the right of the graph of $y = f(x)$.

$y = f(x+1)$



$y = f(x-1)$



Here's a trick I like to use for interpreting horizontal shifts:

For horizontal shifts, the value of x that makes the substituted expression equal to 0 will tell you what the horizontal shift is. For instance, when we have $f(x-1)$, what value of x makes $x-1$ equal to 0? $x=1$. So $f(x-1)$ is 1 unit to the right of $f(x)$. What about $f(x+4)$? Well, $x=-4$ makes $x+4$ equal to 0, so in relation to $f(x)$, $f(x+4)$ is shifted 4 units to the left (negative value = shift to the left). And what about $f(3x-2)$? A value of $x = \frac{2}{3}$ makes $3x-2$ equal to 0, so $f(3x-2)$ is a shift of $\frac{2}{3}$ of a unit to the right of $f(x)$.

Note that horizontal and vertical shifts are commonly referred to as **translations**.

EXAMPLE 1: In the xy-plane, the graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ down 3 units and to the left by 5 units. If the function f is defined by $f(x) = (x-3)^2 + 1$, which equation defines the function g ?

A) $g(x) = (x-8)^2 - 2$ B) $g(x) = (x+2)^2 - 2$ C) $g(x) = (x-8)^2 + 4$ D) $g(x) = (x+2)^2 + 4$

The order in which we apply horizontal and vertical translations does not matter. Applying a shift of 3 units down, we get $f(x) - 3$. Shifting this result to the left by 5 units gives $f(x+5) - 3$ (we substitute $x+5$ for x). Therefore,

$$\begin{aligned} g(x) &= f(x+5) - 3 \\ &= ((x+5)-3)^2 + 1 - 3 = (x+2)^2 - 2 \end{aligned}$$

Answer $\boxed{(B)}$. If you graph this answer, you will see that it is 3 units down from and 5 units to the left of the graph of $f(x)$.

To summarize, for a function $f(x)$,

- $f(x) + a$ results in an upward shift of a units; $f(x) - a$ results in a downward shift of a units
- $f(x+a)$ results in a horizontal shift of a units to the left; $f(x-a)$ results in a horizontal shift of a units to the right

EXERCISE: Answers for this exercise start on page 371.

1

The function f is an exponential function. The y-intercept of the graph of $y = f(x)$ in the xy-plane is $(0, 5)$. What is the y-intercept of the graph of $y = f(x) - 8$?

A) $(-8, 5)$
B) $(8, 5)$
C) $(0, -3)$
D) $(0, 13)$

2

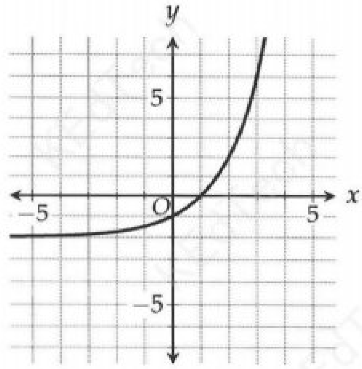
In the xy-plane, which of the following translations of the graph of $y = 2x^2 - 2$ results in the graph of $y = 2x^2 + 4$?

A) A translation 2 units downward
B) A translation 6 units upward
C) A translation 2 units to the left
D) A translation 6 units to the right

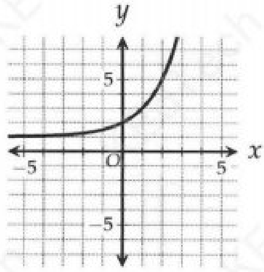
3

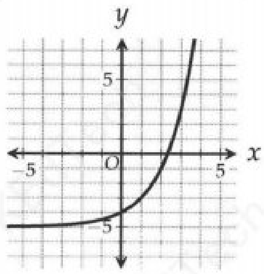
The function f is defined by $f(x) = -(x-5)^2$. In the xy-plane, the graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ up 8 units. What is the y-coordinate of the y-intercept of the graph of $y = g(x)$?

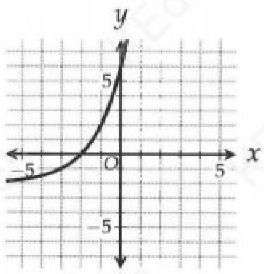
4



The graph of $y = g(x)$ is shown. What is the graph of $y = g(x) + 3$?

A)


B)


C)


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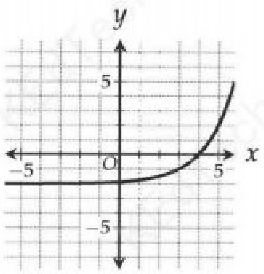

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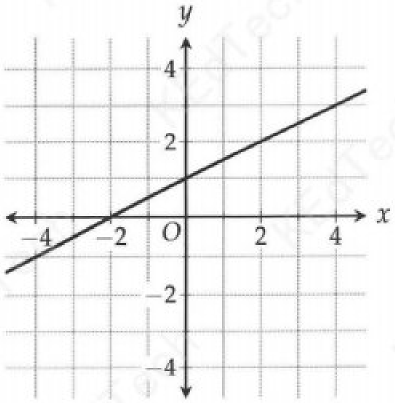


The graph of $y = h(x) + 9$ is shown. Which equation defines function h ?

A) $h(x) = 2x - 8$

B) $h(x) = \frac{1}{2}x - 8$

C) $h(x) = 2x + 10$

D) $h(x) = \frac{1}{2}x + 10$

6

The function f is defined by $f(x) = \frac{x+1}{2}$. In the xy-plane, the graph of $y = g(x)$ is the result of shifting the graph of $y = f(x)$ up 7 units. Which equation defines function g ?

A) $g(x) = \frac{x-6}{2}$

B) $g(x) = \frac{x+8}{2}$

C) $g(x) = \frac{x+15}{2}$

D) $g(x) = \frac{7x+7}{2}$

7

The graph of $y = -x + 3$ is translated 20 units to the right in the xy-plane. Which equation represents the resulting graph?

A) $y = -x - 17$

B) $y = -x + 17$

C) $y = -x + 23$

D) $y = -20x + 3$

8

The function f is defined by $f(x) = ax(x - 7)$, where a is a constant. The graph of $y = f(x)$ in the xy-plane is translated 11 units to the left to produce the graph of $y = g(x)$. Which equation defines function g ?

A) $g(x) = a(x + 11)(x - 7)$

B) $g(x) = a(x + 11)(x + 4)$

C) $g(x) = a(x - 11)(x - 7)$

D) $g(x) = a(x - 11)(x - 18)$

9

The function f is defined by $f(x) = \frac{2}{3}x - 6$. The graph of $y = f(x)$ in the xy-plane is translated 3 units to the left. What is the y-intercept of the resulting graph?

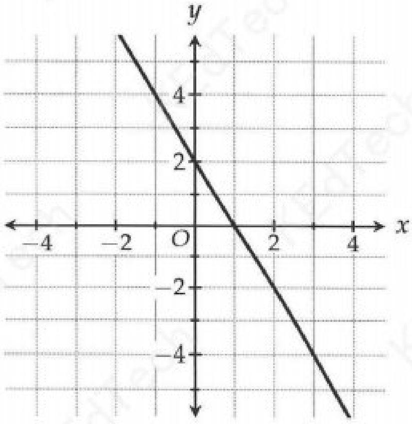
A) $(0, -3)$

B) $(0, -4)$

C) $(0, -8)$

D) $(0, -9)$

10



The graph of $y = f(x)$ is shown in the xy-plane. The function g is defined by $g(x) = f(x) + 4$. What is the x-intercept of the graph of $y = g(x)$?

A) $(-1, 0)$
B) $(2, 0)$
C) $(3, 0)$
D) $(5, 0)$

11

In the xy-plane, the graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ to the right by $1\frac{1}{2}$ units. Which of the following equations defines $g(x)$?

A) $g(x) = f(3x - 2)$
B) $g(x) = f(3x + 2)$
C) $g(x) = f(2x - 3)$
D) $g(x) = f(2x + 3)$

12

In the xy-plane, the graph of $y = f(x)$ is a line that passes through the points $(-1, 82)$ and $(3, 46)$. The graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ down 14 units. For what value of x does $g(x) = 0$?

13

$$g(x) = 3^x - 5$$

The exponential function g is defined by the given equation. The graph of $y = g(x)$ in the xy-plane is translated 2 units down and 3 units to the left to produce the graph of $y = h(x)$. What is the value of $h(0)$?

14

$$f(x) = -2 + \frac{k+6}{x}$$

In the given function, k is a constant. The graph of $y = f(x)$ in the xy-plane is translated 4 units up and 5 units to the right to produce the graph of $y = g(x)$. Which equation defines function g ?

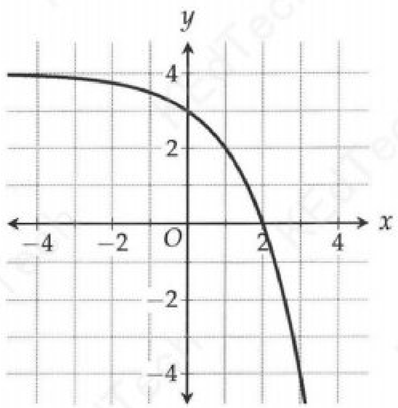
A) $g(x) = -2 + \dfrac{k+10}{x-5}$

B) $g(x) = -2 + \dfrac{k+10}{x+5}$

C) $g(x) = 2 + \dfrac{k+6}{x-5}$

D) $g(x) = 2 + \dfrac{k+6}{x+5}$

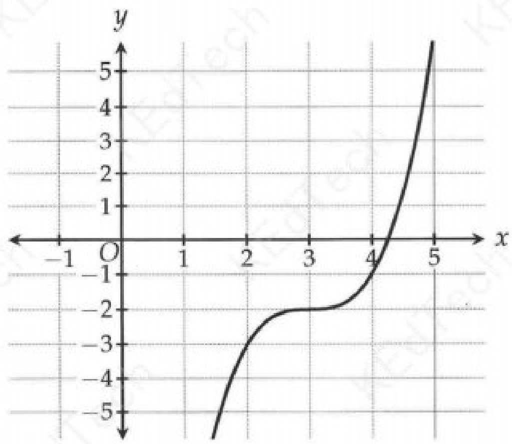
15



The graph of $y = f(x) - 2$ is shown. Which equation defines the function f ?

A) $f(x) = -2^x + 4$
B) $f(x) = -2^x + 5$
C) $f(x) = -2^x + 6$
D) $f(x) = -3^x + 6$

16



The graph of $y = g(x)$ is shown, where the function g is defined by $g(x) = f(x + a) + b$ and a and b are constants. If the function f is defined by $f(x) = x^3$, what is the value of $a + b$?

A) -5
B) -1
C) 1
D) 5

17

The function f is defined by $f(x) = (x - 3)^2$, and the function g is defined by $g(x) = x^2 + 4x + 4$. In the xy-plane, the graph of $y = g(x)$ is the result of shifting the graph of $y = f(x)$ to the left by k units. What is the value of k ?

18

Line p is defined by $12x + 5y = 56$. Line q is the result of translating line p down 6 units in the xy-plane. What is the x-coordinate of the x-intercept of line q ?