1

Exponents & Radicals

Given the expression x^n, x is called the **base** and n is called the **exponent** or **power**. Here are the laws of exponents you should know:

Law	Example
$x^1 = x$	$3^1 = 3$
$x^0 = 1$	$3^0 = 1$
$x^m \cdot x^n = x^{m+n}$	$3^4 \cdot 3^5 = 3^9$
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{3^7}{3^3} = 3^4$
$(x^m)^n = x^{mn}$	$(3^2)^4 = 3^8$
$(xy)^m = x^m y^m$	$(2 \cdot 3)^3 = 2^3 \cdot 3^3$
$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$	$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$
$x^{-m} = \frac{1}{x^m}$	$3^{-4} = \frac{1}{3^4}$

The difference between

$$(-3)^2 \text{ and } -3^2$$

comes down to order of operations (PEMDAS), which dictates that parentheses are prioritized first. So,

$$(-3)^2 = (-3) \cdot (-3) = 9$$

Without parentheses, exponents take priority:

$$-3^2 = -3 \cdot 3 = -9$$

Notice that the negative is not applied until the exponent operation is carried through.
Sometimes, the result turns out to be the same, as in:

$$(-2)^3 \text{ and } -2^3$$

Both yield -8.

EXAMPLE 1: Which of the following is equivalent to $-x^2(-x)^5$?

A) $-x^{10}$ B) $-x^7$ C) x^7 D) x^{10}

The $-x^2$ term cannot be simplified any further because the negative is not inside any parentheses. The $(-x)^5$ term has an odd exponent. An odd exponent yields an odd number of negative signs, which in turn yields a negative result. So, $(-x)^5$ simplifies to $-x^5$. The entire expression then boils down to

$$-x^2(-x)^5 = -x^2 \cdot -x^5 = x^{2+5} = x^7$$

Answer $\boxed{(C)}$.

EXAMPLE 2: Which expression is equivalent to $4^{x+1} \cdot 3^{2x}$, where $x > 0$?

A) $4(12)^x$ B) $4(24)^x$ C) $4(36)^x$ D) $8(12)^x$

We can use our laws of exponents to simplify the expression. First, the law $x^{m+n} = x^m \cdot x^n$ allows us to simplify 4^{x+1} to $4^x \cdot 4^1$. Then the law $x^{mn} = (x^m)^n$ allows us to turn 3^{2x} into $(3^2)^x = 9^x$. After applying these laws, we get

$$4^{x+1} \cdot 3^{2x} = 4^x \cdot 4^1 \cdot 9^x = 4 \cdot (4^x \cdot 9^x) = 4(36)^x$$

Answer $\boxed{(C)}$. In the last step, we used the law $x^m y^m = (xy)^m$ to get $4^x \cdot 9^x = 36^x$.

Roots are the opposites of exponents. Whereas 5^2 will give you 25, the square root of 25, denoted by $\sqrt{25}$, will give you back the 5. Similarly, taking the cube root of 6^3 gives $\sqrt[3]{6^3} = 6$ and taking the fourth root of b^4 gives $\sqrt[4]{b^4} = b$. Notice how the operations cancel each other out.

Instead of using the radical sign, $\sqrt[n]{}$, we can also denote the nth-root using the equivalent fractional exponent $\frac{1}{n}$. For example,

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

What about $x^{\frac{2}{3}}$? The 2 on top means to square x. The 3 on the bottom means to cube root it:

$$\sqrt[3]{x^2}$$

We can see this more clearly if we break it down using our laws of exponents:

$$x^{\frac{2}{3}} = \left(x^2\right)^{\frac{1}{3}} = \sqrt[3]{x^2}$$

The order in which we do the squaring and the cube-rooting doesn't matter. We could've broken it down this way as well:

$$x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2$$

You'll see $\sqrt[3]{x^2}$ more often than $(\sqrt[3]{x})^2$ because the outside cube root avoids the need for parentheses.

EXAMPLE 3: Which of the following is equivalent to $\sqrt[4]{x^5}$?

A) x B) $x^5 - x^4$ C) $x^{\frac{5}{4}}$ D) $x^{\frac{4}{5}}$

The fourth root equates to a fractional exponent of $\frac{1}{4}$, so

$$\sqrt[4]{x^5} = \left(x^5\right)^{\frac{1}{4}} = x^{\frac{5}{4}}$$

Answer $\boxed{\text{(C)}}$.

EXAMPLE 4: The expression $\dfrac{25^{2x}}{\left(\sqrt{5}\right)^x}$ is equivalent to 5^{kx}, where k is a constant. What is the value of k ?

Because most of the laws of exponents involve terms with the same base, an often-used tactic is to convert each term to have the same base. Here, we're asked to simplify to 5^{kx}, so we should convert everything to have the same base of 5. As is common in this type of question, we see that we have a perfect square, 25, which can be written as 5^2. In the denominator, we have $\sqrt{5}$, which can be written as $5^{\frac{1}{2}}$.

$$\frac{25^{2x}}{\left(\sqrt{5}\right)^x} = \frac{(5^2)^{2x}}{(5^{\frac{1}{2}})^x} = \frac{5^{4x}}{5^{\frac{1}{2}x}} = 5^{4x-\frac{1}{2}x} = 5^{\frac{7}{2}x}$$

Therefore, $k = \boxed{\frac{7}{2}}$.

EXAMPLE 5: If the square of a is equal to the cube of b, where $a \geq 0$ and $b \geq 0$, for what value of x is $\sqrt{a^x}$ equal to b ?

Essentially, we need to solve

$$\sqrt{a^x} = b$$

for x given that $a^2 = b^3$. Let's first cube root both sides of $a^2 = b^3$ to isolate b:

$$\sqrt[3]{a^2} = \sqrt[3]{b^3}$$

$$a^{\frac{2}{3}} = b$$

Substituting $a^{\frac{2}{3}}$ for b in the equation above, we get

$$\sqrt{a^x} = a^{\frac{2}{3}}$$

$$a^{\frac{1}{2}x} = a^{\frac{2}{3}}$$

Since both sides have the same base of a, we can just equate the exponents and solve for x:

$$\frac{1}{2}x = \frac{2}{3}$$

$$x = \boxed{\frac{4}{3}}$$

The SAT will also test you on simplifying square roots (also called "surds"). To simplify a square root, factor the number inside the square root and take out any pairs:

$$\sqrt{48} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = \sqrt{\boxed{2 \cdot 2} \cdot \boxed{2 \cdot 2} \cdot 3} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$$

In the example above, we take a 2 out for the first $\boxed{2 \cdot 2}$. Then we take another 2 out for the second pair $\boxed{2 \cdot 2}$. Finally, we multiply the two 2's outside the square root to get 4 and leave the 3 inside. Of course, a quicker route would have been to break down 48 into bigger factor pairs:

$$\sqrt{48} = \sqrt{\boxed{4 \cdot 4} \cdot 3} = 4\sqrt{3}$$

Here's another example:

$$\sqrt{72} = \sqrt{\boxed{2 \cdot 2} \cdot \boxed{3 \cdot 3} \cdot 2} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}$$

To go backwards, take the number outside and put it back under the square root as a pair:

$$6\sqrt{2} = \sqrt{6 \cdot 6 \cdot 2} = \sqrt{72}$$

To simplify a cube root such as $\sqrt[3]{16}$, take out any triplets:

$$\sqrt[3]{16} = \sqrt[3]{\boxed{2 \cdot 2 \cdot 2} \cdot 2} = 2\sqrt[3]{2}$$

EXAMPLE 6: Which of the following is equivalent to $\left(x^2\right)^{\frac{3}{4}}$, where $x > 0$?

A) $\sqrt{x}$ B) $x\sqrt{x}$ C) $\sqrt[3]{x^2}$ D) $\sqrt[4]{x}$

Solution 1:

$$\left(x^2\right)^{\frac{3}{4}} = x^{(2\cdot\frac{3}{4})} = x^{\frac{3}{2}} = \sqrt{x^3} = \sqrt{\boxed{x\cdot x}\cdot x} = x\sqrt{x}$$

Answer $\boxed{(B)}$.

Solution 2: Since $\left(x^2\right)^{\frac{3}{4}} = x^{(2\cdot\frac{3}{4})} = x^{\frac{3}{2}}$, we can compare this exponent of $\dfrac{3}{2}$ to the exponent of x in each of the answer choices.

Choice A: $\sqrt{x} = x^{\frac{1}{2}}$

Choice B: $x\sqrt{x} = x^1 \cdot x^{\frac{1}{2}} = x^{(1+\frac{1}{2})} = x^{\frac{3}{2}}$

Choice C: $\sqrt[3]{x^2} = x^{\frac{2}{3}}$

Choice D: $\sqrt[4]{x} = x^{\frac{1}{4}}$

These results confirm that the answer is $\boxed{(B)}$.

EXERCISE 1: Evaluate WITHOUT a calculator. Answers for this exercise start on page 311.

1. $(-1)^{10}$
2. $(-1)^{15}$
3. $(-1)^8$
4. -1^8
5. $-(-1)^8$
6. $(-3)^3$
7. -3^3
8. $-(-6)^2$
9. $-(-4)^3$
10. $2^3 \times 3^2 \times (-1)^5$
11. 4^{-1}
12. 5^0
13. 3^{-2}
14. $\left(\frac{1}{2}\right)^{-1}$
15. $\left(\frac{4}{3}\right)^{-2}$

EXERCISE 2: Simplify so that your answer contains only positive exponents. The first one has been done for you. Answers for this exercise start on page 312.

1. $2k^{-4} \cdot 4k^2 = \dfrac{2 \cdot 4k^2}{k^4} = \dfrac{8}{k^2}$
2. $3x^2 \cdot 2x^3$
3. $5x^4 \cdot 3x^{-2}$
4. $7m^3 \cdot -3m^{-3}$
5. $(2x^2)^{-3}$
6. $-3a^2b^{-3} \cdot 3a^{-5}b^8$
7. $\dfrac{3x^4}{(x^{-2})^2}$
8. $\dfrac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$
9. $(2m)^2 \cdot (3m^3)^2$
10. $(a^{-1} \cdot a^{-2})^2$
11. $(b^{-2})^{-3} \cdot (b^3)^2$
12. $\dfrac{(m^2n)^3}{(mn^2)^2}$
13. $\dfrac{mn}{m^2n^3}$
14. $\dfrac{k^{-2}}{k^{-3}}$
15. $\dfrac{x^2y^3z^4}{x^{-3}y^{-4}z^{-5}}$

EXERCISE 3: Simplify the radicals or solve for x. Answers for this exercise start on page 312.

1. $\sqrt{12}$
2. $\sqrt{96}$
3. $\sqrt{45}$
4. $\sqrt{18}$
5. $3\sqrt{75}$
6. $\sqrt{32}$
7. $5\sqrt{2} = \sqrt{x}$
8. $3\sqrt{x} = \sqrt{45}$
9. $2\sqrt{2} = \sqrt{4x}$
10. $4\sqrt{6} = 2\sqrt{3x}$
11. $3\sqrt{8} = x\sqrt{2}$
12. $x\sqrt{x} = \sqrt{216}$

EXERCISE 4: Answers for this exercise start on page 313.

1

Which expression is equivalent to $(x^2y)(x^4y^{-3})$, where x, y, and z are positive numbers?

A) x^6y^{-3}

B) x^6y^{-2}

C) x^8y^{-3}

D) x^8y^{-2}

2

Which of the following is equivalent to $\sqrt{\dfrac{x}{64}}$ for all $x > 0$?

A) $\dfrac{x^2}{8}$

B) $\dfrac{x^2}{32}$

C) $\dfrac{x^{\frac{1}{2}}}{8}$

D) $\dfrac{x^{\frac{1}{2}}}{32}$

3

Which expression is equivalent to $\dfrac{x^{\frac{5}{6}}}{\sqrt[3]{x}}$, where $x \neq 0$?

A) $x^{\frac{1}{2}}$

B) $x^{\frac{5}{2}}$

C) $x^{\frac{2}{3}}$

D) $x^{\frac{5}{3}}$

4

Which expression is equivalent to $b^{\frac{9}{7}}$, where $b > 0$?

A) $\sqrt[15]{b^{105}}$

B) $\sqrt[15]{b^{135}}$

C) $\sqrt[105]{b^{135}}$

D) $\sqrt[135]{b^{105}}$

5

Which expression represents the product of $\left(b^6c^{-2}d^{-5}\right)$ and $\left(b^8c^{-3}+c^4d^5\right)$, where b, c, and d are positive?

A) $b^{14}c^{-5}+c^2d^{-10}$

B) $b^{14}c^{-5}+c^2$

C) $b^{14}c^{-5}d^{-5}+b^6c^2$

D) $b^{14}c^{-5}d^{-5}+c^2$

6

If $x \neq 0$ and $y \neq 0$, which of the following is equivalent to $\dfrac{8x^2}{\sqrt{4x^6y^4}}$?

A) $2xy^{-2}$

B) $4x^{-2}y^2$

C) $4x^{-1}y^{-2}$

D) $4xy^2$

7

If r and s are positive, which of the following expressions is equivalent to $r^{\frac{6}{7}}s^{\frac{3}{7}}$?

A) $\dfrac{1}{\sqrt[6]{r^7s^{14}}}$

B) $\sqrt[6]{r^7s^{14}}$

C) $\dfrac{1}{\sqrt[7]{r^6s^3}}$

D) $\sqrt[7]{r^6s^3}$

8

$$\sqrt[b]{a} = \sqrt[d]{c}$$

The given equation relates the distinct positive numbers a, b, c, and d. Which equation correctly expresses c in terms of a, b, and d ?

A) $c = \dfrac{d}{a^{\frac{1}{b}}}$

B) $c = a^{d-b}$

C) $c = a^{\frac{b}{d}}$

D) $c = a^{\frac{d}{b}}$

9

Which expression is equivalent to $\sqrt{16a^{\frac{2}{3}}}$, where $a > 0$?

A) $4a^{\frac{1}{3}}$

B) $4a^{\frac{4}{3}}$

C) $8a^{\frac{1}{3}}$

D) $8a^{\frac{4}{3}}$

10

Which expression is equivalent to $\sqrt{rs}\,(\sqrt{r} + \sqrt{s})$, where $r \geq 0$ and $s \geq 0$?

A) $\sqrt{r^2s + rs^2}$

B) $r\sqrt{s} + s\sqrt{r}$

C) $rs\sqrt{r+s}$

D) $\sqrt{rs + r + s}$

11

The expression $\left(\sqrt[3]{x^2}\right)^n$, where n is a constant, is equivalent to x^8. What is the value of n ?

12

Which expression is equivalent to $2^{-2k} \cdot 3^k$?

A) $(3\sqrt{2})^k$

B) $\left(\dfrac{1}{36}\right)^k$

C) $\left(\dfrac{3}{4}\right)^k$

D) $\left(\dfrac{9}{4}\right)^k$

13

Which expression is equivalent to $a^{\frac{2}{9}}(a^{\frac{2}{3}})^{\frac{2}{3}}$, where a is positive?

A) $\sqrt[3]{a^2}$

B) $\sqrt[3]{a^{16}}$

C) $\sqrt[9]{a^8}$

D) $\sqrt[9]{a^{14}}$

14

Which expression is equivalent to $\frac{4}{9}\left(\frac{2}{3}\right)^a$?

A) $\left(\frac{2}{3}\right)^{\frac{a}{2}}$

B) $\left(\frac{2}{3}\right)^{a-2}$

C) $\left(\frac{2}{3}\right)^{a+2}$

D) $\left(\frac{2}{3}\right)^{2a}$

15

$$\sqrt[4]{x^3} \cdot \sqrt[12]{x^5}$$

The given expression is equivalent to $\sqrt[m]{x^n}$, where m and n are positive constants each less than 10 and $x > 1$. What is the value of $\frac{m}{n}$?

16

Two numbers, p and q, are each greater than zero, and the square of p is equal to the cube root of q. For what value of x is p^{1-x} equal to the square root of q ?

EXERCISE 5: Answers for this exercise start on page 314.

1

Which expression is equivalent to $\sqrt{100x^{36}}$?

A) $10x^6$
B) $50x^6$
C) $10x^{18}$
D) $50x^{18}$

2

If $(5^3)^{4k} = (5^{\frac{1}{3}})^{24}$, what is the value of k ?

A) -6
B) $\frac{2}{3}$
C) $\frac{3}{4}$
D) 2

3

Which of the following is equivalent to $3y^{\frac{1}{2}}$ for all $y > 0$?

A) $\sqrt{3y}$
B) $\sqrt{9y}$
C) $\sqrt{\frac{3}{y}}$
D) $\sqrt{\frac{9}{y}}$

4

Which expression is equivalent to 4^{m+2} ?

A) 16^m
B) $16+4^m$
C) $8(4^m)$
D) $16(4^m)$

5

Which expression is equivalent to $\sqrt[4]{x^2y^4}$, where $x > 0$ and $y > 0$?

A) $\sqrt{xy}$
B) $y\sqrt{x}$
C) $\frac{1}{x^2}$
D) x^2y

6

Which expression is equivalent to $p^{\frac{1}{3}} \cdot \left(\sqrt[3]{p}\right)^2$, where p is greater than 0 ?

A) $p^{\frac{2}{3}}$
B) $p^{\frac{2}{9}}$
C) $p^{\frac{7}{3}}$
D) p

7

Which expression is equivalent to $x^{\frac{2a}{b}}$ for all positive values of x, where a and b are positive integers?

A) $\sqrt[b]{ax^2}$
B) $\sqrt[b]{x^{2a}}$
C) $\sqrt[b]{x^{a+2}}$
D) $\sqrt[2a]{x^b}$

8

If $\frac{9^7}{\sqrt[4]{9^{10}}} = 9^{6t}$, what is the value of t ?

9

Which expression is equivalent to $a^{\frac{3}{4}}b^{\frac{1}{2}}$, where $a \geq 0$ and $b \geq 0$?

A) $\sqrt[4]{a^3b}$

B) $\sqrt[4]{a^3b^2}$

C) $\sqrt{a^3b}$

D) $\sqrt{a^4b^2}$

10

Which expression is equivalent to $h^{\frac{5}{12}}(h^{-\frac{1}{4}})^{\frac{7}{3}}$, where $h > 0$?

A) $\dfrac{1}{h^6}$

B) $\sqrt{h^5}$

C) $\dfrac{1}{\sqrt[6]{h}}$

D) $\dfrac{1}{\sqrt[5]{h^2}}$

11

$$\sqrt[8]{41k}\left(\sqrt[9]{41k}\right)^{12}$$

For what value of x is the given expression equivalent to $(41k)^{25x}$, where $k > 1$?

12

The expression 4^{18x} is equivalent to k^{3x}, where k is a constant. What is the value of k ?

13

Which expression is equivalent to $9^{\frac{1}{n}}(4^{\frac{1}{2n}})$, where n is a positive integer?

A) $24^{\frac{1}{n}}$

B) $12^{\frac{1}{n}}$

C) $\sqrt[n]{18}$

D) $\sqrt[n]{6}$

14

$$\frac{10\sqrt[6]{9^3x^{48}}}{\sqrt[4]{6^4x}}$$

The given expression is equivalent to ax^b, where $a > 0$, $b > 0$, and $x > 1$. What is the value of ab ?

15

If m and n are both positive numbers, and $4m$ is equal to the cube root of the square of n, for what value of x is m^x equal to n when $m = 2$?

16

If $2^{x+3} - 2^x = k(2^x)$, what is the value of k ?

A) 3

B) 5

C) 7

D) 8